Georgia Tech Center for Academic Success

MATH 2550/2551 READING DAY STUDY SESSION WORKSHEET

Problems marked with ** are *only* relevant for MATH 2551.

Problems

1. Find T, N, and curvature for $r(t) = (3\sin t)\vec{i} + (3\cos t)\vec{j} + 4t\vec{k}$.
2. **Write acceleration in terms of tangential and normal components for $r(t) = (t+1)\vec{i} + 2t\vec{j} + t^2\vec{k}$, $t = 1$.
3. Find the limit: $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$.
4. Find all second-order partial derivatives for $w = x\sin(x^2 y)$.
5. Evaluate $\frac{dw}{dt}$ for $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = e^t$, $t = 1$.
6. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ for $z^3 - xy + yz + y^3 - 2 = 0$, $(1,1,1)$.
7. Find the derivative of the function at the point in the direction of $\vec{v}$ for $f(x,y) = 2xy - 3y^2$, $P = (5,5)$, $\vec{v} = 4\vec{i} + 3\vec{j}$.
8. Find the tangent plane and normal line for $2z - x^2 = 0$, $P_0(2,0,2)$.
9. Find all local maxima, minima, and saddle points for $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$.
10. Find absolute maxima and minima for $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.
11. Find the maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.
12. Compute the area bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$ by an iterated integral using (a) vertical cross-sections and (b) horizontal cross-sections.
13. Sketch the region and evaluate $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$.
14. Change to polar and evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy\,dx$.
15. For the region below $z = 4 - xy$ and in $0 \le x \le 2$, $0 \le y \le 1$, integrate the function $3 - 4x$.
16. Use $x = \frac{u}{v}$, $y = uv$ in $\mathbb{R}$ (first quadrant) bounded by $xy = 1$, $xy = 9$, $y = x$, $y = 4x$ for $\displaystyle\iint \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy$.
17. **Evaluate the line integral $\int (xy + y + z)\,ds$ along the curve $r(t) = 2t\vec{i} + t\vec{j} + (2-2t)\vec{k}$, $0 \le t \le 1$.
18. **Find the flow of the field $F = -4xy\vec{i} + y8\vec{j} + 2\vec{k}$ along $r(t) = t\vec{i} + t^2\vec{j} + \vec{k}$, $0 \le t \le 2$.
19. **Find the circulation and flux of the field $F = x\vec{i} + y\vec{j}$ around $r(t) = \cos t\vec{i} + \sin t\vec{j}$, $0 \le t \le 2\pi$.
20. **Find the potential function for $F = e^{y+2z}(\vec{i} + x\vec{j} + 2x\vec{k})$.

21. **Use Green's Theorem to find counterclockwise circulation and outward flux for field $F = (y^2 - x^2)\vec{i} + (x^2 + y^2)\vec{j}$ and curve C: $y = 0$, $x = 3$, $y = x$.

22. **Use a parametrization to express the area of the surface as a double integral: the portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$.

23. Evaluate $\iint 2y\, dV$ over $S : y^2 + z^2 = 4$ between $x = 0$, $x = 3 - z$.

24. At time $t = 0$, a particle is located at the point $(0, 1, 2)$. At this time it is traveling towards the point $(3, 2, 3)$, has speed 3 at $(0, 1, 2)$, and has constant acceleration $3\vec{i} - \vec{j} + \vec{k}$. Find an equation for the position vector $\vec{r}(t)$ at time t.

25. Let $P(A_0, B_0, C_0)$, $Q(A_1, B_1, C_1)$ be two distinct points in 3-dimensional space.
   (a) Write down a vector parameterization for the line PQ.
   (b) Show that the curvature κ of PQ is 0.
   (c) **Show that the torsion τ of PQ is 0.

26. (a) At what points (x, y) in the plane is the function $f(x, y) = \frac{y}{1+\cos x}$ continuous?
   (b) Find the following limit by first rewriting the fraction $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^3 + y^3}{x + y}$.

27. Let C be the smooth curve given by the intersection of the two surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$.
   (a) Write down an equation that describes C implicitly.
   (b) Write down a parametric equation for the line tangent to C at the point $(1, 1, 1)$.

28. Let D be the cylinder bounded below by $z = -1$, bounded on the sides by $x^2 + y^2 = 1$, and bounded above by $z = 1$. (**Do not evaluate the integrals.**)
   (a) Express the volume of D as an iterated triple integral in cylindrical coordinates.
   (b) Express the volume of D as an iterated triple integral in rectangular coordinates.
   (c) Express the volume of D as an iterated triple integral in spherical coordinates.

29. Consider the function $f(x, y) = \dfrac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$.
   (a) Describe the region R given by $1 \le x^2 + y^2 \le e$ in polar coordinates.
   (b) Write down the integral of $f(x, y)$ over R in polar coordinates.
   (c) Evaluate the integral you found in part (b).

30. Let D be the region in xyz-space defined by the inequalities $1 \le x \le 2$, $0 \le xy \le 2$, $0 \le z \le 1$. Consider the coordinate transformation of D to the uvw-plane given by $u = x$, $v = xy$, $w = 3z$.
   (a) Sketch the preimage G of D under the coordinate transformation in the uvw-plane and label the bounding curves.
   (b) Write down the Jacobian associated to this coordinate transformation.
   (c) Evaluate the integral $\displaystyle\iiint_D (x^2 y + 3xyz)\,dx\,dy\,dz$.

31. A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. Points on the plate have temperature $T(x, y) = x^2 + 2y^2 - x$. Find the temperatures of the hottest and coldest points of the plate.

32. Find the points on the surface $xyz = 1$ closest to the origin.

33. Find the volume of the wedge cut from the cylinder $x^2 + y^2 \le 1$ by the planes $z = -y$ and $z = 0$.

34. Consider $I = \int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx$.
Rewrite the integral as an equivalent iterated integral in the order $dy\, dx\, dz$

35. $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$. C is the triangle bounded by $y=0$, $x=1$, $y=2x$. Find the counterclockwise circulation for $\underset{\sim}{F}$ around C. Find the outward flux for $\mathbf{F}$ over C.

36. **Find the surface area of the cap cut from the paraboloid $x^2+z^2=3y$ by the plane $y=1$.

37. Use double integral to find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$ via polar coordinates.

38. Find the center of mass of a solid of constant density that is bounded by $x=y^2$ and the planes $x=z$,$z=0$ and $x=1$. Sketch the solid.

39. Use Taylor's formula to find the quadratic and cubic approximation of $f(x,y)=e^x \sin y$ near the origin. Also, estimate the error in the quadratic approximation if $|x| \le 0.1$ and $|y| \le 0.1$

40. A man throws a rock at an angle of 45° to the horizontal at an initial speed of $25\sqrt{2}$ ft/s. It leaves his hand 20 ft above the ground. There is a 7 ft tall board 50 ft away.
(a) Where is the rock 1 seconds later?
(b) How high does the rock go?
(c) Does the rock hit the board?

41. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

42. Use the surface integral in Stokes' Theorem to calculate the circulation of field $\vec{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$ around the curve C, where C is the intersection of the cylinder $x^2+y^2=4$ and the hemisphere $x^2+y^2+z^2=16$, $z \ge 0$, counterclockwise when viewed from above.

43. Use the surface integral in Stoke's Theorem to calculate the flux of the curl of the field $\vec{F} = (x-y)\hat{i} + (y-z)\hat{j} + (z-x)\hat{k}$ across the surface S in the direction of the outward unit normal $\vec{n}$, where S is parametrized by $\vec{r}(r,\theta) = (r\cos\theta)\hat{i} + (r\sin\theta)\hat{j} + (5-r)\hat{k}$, $0 \le r \le 5$,$0 \le \theta \le 2\pi$.

44. Use the Divergence Theorom to find the outward flux of $\mathbf{F}$ across the boundary of the region D. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
(a) D is the cube cut from the first octant by the planes $x=1$, $y=1$, and $z=1$.
(b) D is the region cut from the solid cylinder $x^2+y^2 \le 4$ by the planes $z=0$ and $z=1$.

Answers

1. $T = \frac{3\cos t}{5}\vec{i} - \frac{3\sin t}{5}\vec{j} + \frac{4}{5}\vec{k}$, $N = (-\sin t)\vec{i} - (\cos t)\vec{j}$, $\kappa = \frac{3}{25}$

2. $a(1) = \frac{4}{3}T + \frac{2\sqrt{5}}{3}N$

3. $\frac{5}{2}$

4. $w_{xx} = 6xy\cos(x^2y) - 4x^3y^2\sin(x^2y)$, $w_{yy} = -x^5\sin(x^2y)$, $w_{xy} = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$

5. $\frac{dw}{dt} = 4t\arctan t + 1 = \pi + 1$

6. $\frac{1}{4}, -\frac{3}{4}$

7. -4

8. $2x - z - 2 = 0$, $r(t) = (2,0,2) + t(-4,0,2)$

9. $f(-3,3) = -5$ minimum

10. $(0,0) = 1$, abs max, $(1,2) = -5$, abs min

11. 39

12. (a) Vertical: $\int_0^9 \int_0^{\sqrt{x}} dydx = 18$; (b) Horizontal: $\int_0^3 \int_{y^2}^9 dxdy = 18$

13. $e - 2$

14. $\frac{\pi}{2}$

15. $\int_0^2 \int_0^1 \int_0^{4-xy} (3-4x)dz\,dy\,dx = -\frac{17}{3}$

16. $\int_1^2 \int_1^3 \frac{(u+v)2u}{v} du\,dv = 8 + \frac{52}{3}\ln 2$

17. $\frac{13}{2}$

18. 48

19. Circulation $= 0$; Flux $= 2\pi$

20. $f(x,y,z) = xe^{y+2z} + C$

21. Flux $= -9$, Circulation $= 9$

22. $\int_0^{2\pi} \int_1^3 r\sqrt{5}dr\,d\theta$

23. $r(u,v) = r(x,\theta) = \langle x, 2\sin\theta, 2\cos\theta\rangle$

$$\iint f(r(u,v))|r_u \times r_v|dA = \int_0^{2\pi} \int_0^{3-2\cos\theta} 2(2\sin\theta)(2)dx\,d\theta = 0$$

24. $\vec{r}(t) = \left(\frac{3}{2}t^2 + \frac{9}{\sqrt{11}}t\right)\vec{i} + \left(-\frac{1}{2}t^2 + \frac{3}{\sqrt{11}}t + 1\right)\vec{j} + \left(\frac{1}{2}t^2 + \frac{3}{\sqrt{11}}t + 2\right)\vec{k}$

25. (a)$L(t) = \langle x(t) = A_0 + (A_1 - A_0)t, y(t) = B_0 + (B_1 - B_0)t, z(t) = C_0 + (C_1 - C_0)t\rangle$

26. (a) f is continuous on $\{(x,y)|x \neq (2k+1)\pi; k = \cdots, -1, 0, 1, \cdots\}$. (b) 1

27. (a) $x^2 + 2y^2 + 3z^2 = 6xyz$. (b) $L(t) = \langle 1+2t, 1-4t, 1+2t \rangle$

28. (a) $\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1} r\,dr\,d\theta\,dz$. (b) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-1}^{1} dz\,dx\,dy$
(c) $\int_{0}^{2\pi}\int_{\pi/4}^{3\pi/4}\int_{0}^{1/\sin\varphi} \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta + 2\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{1/\cos\varphi} \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta$

29. (a) $1 \leq r^2 \leq e$. (b) $\int_{0}^{2\pi}\int_{1}^{\sqrt{e}} 2\ln(r)dr\,d\theta$. (c) $-2\pi\sqrt{e} + 4\pi$

30. (b) $\frac{1}{3u}$. (c) $2 + 3\ln(2)$

31. Hottest: $\frac{9}{4}$ at points $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Coldest: $-\frac{1}{4}$ at point $(\frac{1}{2}, 0)$.

32. Points $(1,1,1)$, $(-1,-1,-1)$, $(1,-1,-1)$, and $(-1,1,-1)$ are all at distance 3 from the origin.

33. $\frac{2}{3}$

34. $I = \int_{0}^{1}\int_{0}^{1}\int_{-1}^{-\sqrt{z}}\ dy\,dx\,dz$.

35. circulation $= -7/3$; flux $= -1/3$.

36. $(\pi/6)[7\sqrt{21} - 9]$

37. $\dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$

38. $\left(\dfrac{5}{7}, 0, \dfrac{5}{14}\right)$

39. quadratic approximation: $y + xy$ and cubic approximation: $y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3$, Error $< \frac{e^{0.1}}{750}$

40. (a) $(25, 29)$.
(b) $\dfrac{1905}{64}$ ft.
(c) Yes.

41. 37.2π cm^3.

42. -8π

43. 25π

44. (a) 3; (b) 4π