

**MATH 1553**  
**SAMPLE FINAL EXAM, FALL 2018**

Name	Section

Circle the name of your instructor below:

Bonetto                      Brito 1:55-2:45 PM                      Brito 3:00-3:50 PM  
Duan                              Jankowski                              Kordek  
Margalit 11:15 AM -12:05 PM                      Margalit 12:20-1:10 PM                      Rabinoff  
Srinivasan 3:00-3:50 PM                              Srinivasan 4:30-5:20 PM

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Please show your work. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.



## Problem 1.

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**.

You do not need to justify your answer, and there is no partial credit.

In each case, assume that the entries of all matrices and all vectors are real numbers.

- a) **T** **F** If  $A$  is an  $n \times n$  matrix and  $\text{rank}(A) = 1$ , then every column vector of  $A$  lies on the same line through the origin in  $\mathbf{R}^n$ .

- b) **T** **F** The transformation  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  given below is linear.

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \\ z + 1 \end{pmatrix}.$$

- c) **T** **F** Let  $W = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ . The matrix  $A$  for orthogonal projection onto  $W$  is

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

- d) **T** **F** The least-squares solution to  $Ax = b$  is unique if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

- e) **T** **F** Suppose  $u, v, w$  are vectors in  $\mathbf{R}^n$ . If  $u$  is orthogonal to  $v$  and  $u$  is orthogonal to  $w$ , then  $u$  is orthogonal to  $v - w$ .

## Problem 2.

*Short answer questions:* you need not explain your answers, but show any computations in part (d). In each case, assume that the entries of all matrices are real numbers.

- a) Give an example of a  $3 \times 3$  matrix whose eigenspace corresponding to the eigenvalue  $\lambda = 4$  is a two-dimensional plane.

b) Let  $A = \begin{pmatrix} a & 15 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$ .

$A$  is not invertible when  $a = \underline{\hspace{2cm}}$ .

In this case,  $A$  is / is not diagonalizable (circle one.)

- c) Suppose  $A$  is a  $3 \times 3$  matrix. Which of the following are possible?  
(Circle all that apply.)

- (1) All of its eigenvalues are real, and the matrix is not diagonalizable.
- (2) Its eigenspace corresponding to the eigenvalue  $\lambda = -5$  is a plane, and the algebraic multiplicity of  $-5$  as an eigenvalue is 1.
- (3) Every nonzero vector in  $\mathbf{R}^3$  is an eigenvector of  $A$ .

- d) Find the area of the triangle with vertices  $(-3, 1)$ ,  $(0, 2)$ ,  $(-1, -2)$ .

### Problem 3.

*Short answer questions:* you need not explain your answers. In each case, assume that the entries of all matrices and vectors are real numbers.

a) Which of the following are subspaces of  $\mathbf{R}^3$ ? Circle all that apply.

(1)  $\text{Nul}(A)$ , where  $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix}$ .

(2) The set of solutions to  $T(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , where  $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$ .

(3) The eigenspace corresponding to  $\lambda = 1$ , for any  $3 \times 3$  matrix  $B$  that has 1 as an eigenvalue.

b) Let  $T : \mathbf{R}^4 \rightarrow \mathbf{R}^3$  be a linear transformation with standard matrix  $A$ , so  $T(v) = Av$ . Which of the following are possible? Circle all that apply.

(1) The equation  $Ax = 0$  has only the trivial solution.

(2)  $\text{rank}(A) = \dim(\text{Nul } A)$ .

(3) The equation  $Ax = b$  is consistent for each  $b$  in  $\mathbf{R}^3$ .

c) Suppose  $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -2$ . Find  $\det(3A)$  if  $A = \begin{pmatrix} -4a + d & -4b + e & -4c + f \\ a & b & c \\ g & h & i \end{pmatrix}$ .

d) Let  $v, w$  in  $\mathbf{R}^6$  be orthogonal vectors with  $\|v\| = 2$  and  $\|w\| = 3$ . Let

$$x = 3v - w \quad y = v + w.$$

Find the dot product  $x \cdot y$

## Problem 4.

- a) Let  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the rotation counterclockwise by 90 degrees. Find the standard matrix  $A$  for  $T$  (in other words,  $T(v) = Av$ ).

- b) Let  $U : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  be the linear transformation given by

$$U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - x \\ x + y + z \end{pmatrix}.$$

Find the standard matrix  $B$  for  $U$ .

- c) Compute  $(T \circ U) \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ .

## Problem 5.

Consider the subspace  $V$  of  $\mathbf{R}^4$  given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid x - 2y + 5z = 0 \text{ and } -\frac{z}{2} + w = 0 \right\}.$$

a) Find a basis for  $V$ .

b) Find a basis for  $V^\perp$ .

c) Is there a matrix  $A$  so that  $\text{Col}(A) = V$ ? If so, find such an  $A$ . If not, justify why no such  $A$  exists.

## Problem 6.

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- a) Find the eigenvalues of  $A$ .
- b) Find the eigenspace for each eigenvalue of  $A$ .
- c) Is  $A$  diagonalizable? If your answer is yes, find an invertible  $P$  and a diagonal matrix  $D$  so that  $A = PDP^{-1}$ . If your answer is no, explain why  $A$  is not diagonalizable.



## Problem 7.

Let  $A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}$ .

- a) Find the (complex) eigenvalues of  $A$ . For full credit, you must write your answers in the spaces below.

The eigenvalue with *positive* imaginary part is  $\lambda_1 = \underline{\hspace{2cm}}$ .

The eigenvalue with *negative* imaginary part is  $\lambda_2 = \underline{\hspace{2cm}}$ .

- b) For each of the eigenvalues of  $A$ , find an eigenvector.

For full credit, you must write your answers in the spaces below.

An eigenvector for  $\lambda_1$  (the eigenvalue with *positive* imaginary part) is  $v_1 = \begin{pmatrix} \phantom{0} \\ \phantom{0} \end{pmatrix}$ .

An eigenvector for  $\lambda_2$  (the eigenvalue with *negative* imaginary part) is  $v_2 = \begin{pmatrix} \phantom{0} \\ \phantom{0} \end{pmatrix}$ .

## Problem 8.

Consider an internet with three pages 1, 2, and 3.

- Page 1 links to pages 2 and 3.
- Page 2 links only to page 3.
- Page 3 links to Page 1 and 2.

a) Write the importance matrix  $A$  for this internet.

b) Find the steady-state vector  $v$  for  $A$ .

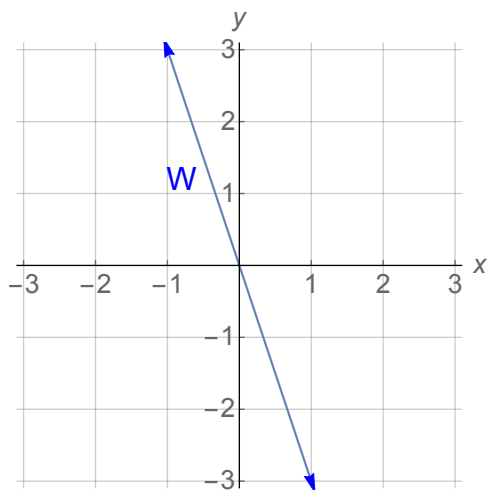
c) Which page has the highest page rank?

## Problem 9.

Let  $W$  be the line  $y = -3x$  in  $\mathbf{R}^2$ , and let  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the linear transformation corresponding to orthogonal projection onto  $W$ .

a) Find the standard matrix  $A$  for  $T$ .

b) Draw  $W^\perp$  below. Be precise!



c) Let  $z = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$ . Find vectors  $z_W$  in  $W$  and  $z_{W^\perp}$  in  $W^\perp$  so that  $z = z_W + z_{W^\perp}$ .

### Problem 10.

Find the least-squares line  $y = Mx + B$  that approximates the data points

$$(-2, -11), \quad (0, -2), \quad (4, 2).$$

**Scratch paper. This sheet will not be graded under any circumstances.**