

MATH 1554 READING DAY STUDY SESSION WORKSHEET

PROBLEMS

1. A 5×4 matrix $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4]$ has all non-zero columns, and $\vec{a}_4 = 2\vec{a}_1 + 3\vec{a}_2 + 5\vec{a}_3$. Find a non-trivial solution to $A\vec{x} = \vec{0}$.

2. For what values of h , if any, are the columns of A linearly dependent? $A = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & 1 \\ h & 1 & 0 \end{bmatrix}$

3. For what values of h is \vec{b} in the plane spanned by \vec{a}_1 and \vec{a}_2 ?

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

4. Express the solution to $A\vec{x} = \vec{0}$ in parametric vector form, where $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

5. Write down the standard matrix A of $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $T(\vec{x}) = -\vec{x}$.

6. Find the domain and co-domain of the linear transformation T given by the standard matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 7 & 3 \\ 2 & 5 & -1 \end{bmatrix}$$

Is this linear transformation one-to-one? Is it onto?

7. Let $A = \begin{bmatrix} -5 & 2 \\ -1 & -3 \end{bmatrix}$. Find its eigenvalue(s) and find an invertible matrix P and a (rotation-scaling) matrix C such that $A = PCP^{-1}$.

8. W is the set of all vectors of the form $\begin{bmatrix} x \\ x+y \\ y \end{bmatrix}$. Which of the following vectors are in W^\perp ?

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

9. Identify all values of a, b, c , if any, so that the columns of U are mutually orthogonal, $U = \begin{bmatrix} 3 & 2 & 2 \\ -4 & 1 & b \\ 2 & a & c \end{bmatrix}$.

10. Use the Gram-Schmidt process to construct an orthonormal basis of the column space of $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ -1 & -4 \end{bmatrix}$.

11. Let $A = QR$, where $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -2 & -2 \end{bmatrix}$, $Q = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}$. Compute the upper triangular matrix R .

12. Give an example of a 2×2 matrix that is in echelon form, is orthogonally diagonalizable, but is not invertible.

13. Make a change of variable, $\vec{x} = P\vec{y}$, that transforms $Q(\vec{x}) = 3x_1^2 + 4x_1x_2$ into a form that does not have cross-product terms. Give P and the new quadratic form.

14. Construct a SVD of matrix $A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

15. True or False?

- (i) If the set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, so is every pair of vectors $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{v}, \vec{w}\}$. (xviii) Every real 3×3 matrix must have a real eigenvalue.
- (ii) If every pair of vectors $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{v}, \vec{w}\}$ is linearly independent, so is the set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$. (xix) For any three vectors \vec{x} , \vec{y} , and \vec{z} we have $(\vec{x} \cdot \vec{y})\vec{z} = (\vec{y} \cdot \vec{z})\vec{x}$.
- (iii) For any two vectors \vec{u} and \vec{v} , we have $Span\{\vec{u}, \vec{v}\} = Span\{\vec{u}, 2\vec{u} + 3\vec{v}, 4\vec{v}\}$. (xx) Let $\vec{x} \cdot \vec{y} > 0$. Then the angle between \vec{x} and \vec{y} is less than 90° .
- (iv) If \vec{u} and \vec{v} are two distinct nonzero vectors, then there are exactly two vectors in $Span\{\vec{u}, \vec{v}\}$. (xxi) Every orthogonal set of nonzero vectors $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.
- (v) The transformation given by $T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 x_2 \\ x_2 \end{bmatrix}$ is linear. (xxii) Let \hat{y} be the orthogonal projection of a vector \vec{y} onto the subspace $W \subset \mathbb{R}^n$. Then the transformation $T(\vec{y}) = \hat{y}$ is linear.
- (vi) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a projection onto the x_1 -axis. The range of T is \mathbb{R}^2 . (xxiii) The inverse of an orthogonal matrix Q is Q^T .
- (vii) The transformation given by $T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix}$ is linear. (xxiv) A least square solution \hat{x} to $A\vec{x} = \vec{b}$ always satisfies $A\vec{x} = \vec{b}$.
- (viii) A linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ can be onto. (xxv) A least square solution \hat{x} to $A\vec{x} = \vec{b}$ minimizes the distance $\|A\vec{x} - \vec{b}\|$. That is, the distance is the shortest for $\vec{x} = \hat{x}$.
- (ix) The composition $S \circ T$ of two one-to-one linear maps is one-to-one. (xxvi) A $n \times n$ symmetric matrix A will always have n real and distinct eigenvalues.
- (x) The range of a one-to-one linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ may be a line. (xxvii) A $n \times n$ symmetric matrix A will have algebraic multiplicity = geometric multiplicity for each of its eigenvalues.
- (xi) The eigenvalues of a square matrix A are the same as the eigenvalues of its reduced row echelon form. (xxviii) If a matrix A is orthogonally diagonalizable, then A^k is also orthogonally diagonalizable for all $k \in \mathbb{Z}^+$.
- (xii) If \vec{u} and \vec{v} are eigenvectors corresponding to the same eigenvalue λ , then every linear combination of $a\vec{u} + b\vec{v}$ with $a, b \in \mathbb{R}$ (except the zero vector) is an eigenvector. (xxix) The eigenvalues of $A^T A$ are always real for any $m \times n$ matrix A .
- (xiii) The geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity. (xxx) A negative definite matrix cannot be invertible.
- (xiv) All upper triangular 3×3 stochastic matrices are not regular. (xxxii) For any matrix A , $A^t A$ has non-negative, real eigenvalues.
- (xv) If A is a diagonalizable matrix, then $\lambda = 0$ is not an eigenvalue of A . (xxxiii) The maximum value of the quadratic form, $Q = 8x_1^2 + 4x_2^2 + x_3^2$, for any $\vec{x} \in \mathbb{R}^3$, is 8.
- (xvi) An $n \times n$ matrix with n distinct eigenvalues is diagonalizable. (xxxiv) If the number of non-zero singular values of a square matrix A equals the number of its columns, then A is invertible.
- (xvii) If complex λ is an eigenvalue, then so is $-\lambda$. (xxxv) If A is an orthogonal matrix, then the largest singular value of A is 1.

ANSWERS

1. $\vec{x} = (2, 3, 5, -1)$

2. None. The columns of A are linearly independent.

3. $h = 3$

4. $\vec{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$

5. $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

6. Domain: \mathbb{R}^3 , Codomain: \mathbb{R}^4 , Not one-to-one, Not onto.

7. $\lambda = -4 \pm i, P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} -4 & -1 \\ 1 & -4 \end{bmatrix}$

8. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

9. $a = -1, b = 7, c = 11$

10. $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}, \text{col}(A) = \text{Span} \left\{ \frac{\vec{v}_1}{\sqrt{6}}, \frac{\vec{v}_2}{2\sqrt{2}} \right\}$

11. $R = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

13. Eigenvalues are $-1, 4, P = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, Q = 4y_1^2 - y_2^2.$

14. $U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right)^T$

15. True/false:

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|--------------|--------------|--------------------------------------------------------------|
| (i) True | (xiii) True | (xxv) True |
| (ii) False | (xiv) True | (xxvi) False |
| (iii) True | (xv) False | (xxvii) True |
| (iv) False | (xvi) True | (xxviii) True |
| (v) False | (xvii) False | (xxix) True |
| (vi) False | (xviii) True | (xxx) False |
| (vii) False | (xix) False | (xxxi) True |
| (viii) False | (xx) True | (xxxii) True |
| (ix) True | (xxi) True | (xxxiii) False |
| (x) False | (xxii) True | (xxxiv) True |
| (xi) False | (xxiii) True | (xxxv) True: $\sigma_1 = \ A\vec{v}_1\ = \ \vec{v}_1\ = 1$ |
| (xii) True | (xxiv) False | |