MATH 2550/2551 READING DAY STUDY SESSION WORKSHEET

Problems marked with ** are *only* relevant for MATH 2551.

Problems

1. Find T, N, and curvature for $r(t) = (3\sin t)\vec{i} + (3\cos t)\vec{j} + 4t\vec{k}$.
2. **Write acceleration in terms of tangential and normal components for $r(t) = (t+1)\vec{i} + 2t\vec{j} + t^2\vec{k}$, $t = 1$.
3. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$.
4. Find all second-order partial derivatives for $w = x\sin(x^2 y)$.
5. Evaluate $\frac{dw}{dt}$ for $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \arctan t$, $z = e^t$, $t = 1$.
6. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ for $z^3 - xy + yz + y^3 - 2 = 0$, $(1,1,1)$.
7. Find the derivative of the function at the point in the direction of $\vec{v}$ for $f(x,y) = 2xy - 3y^2$, $P = (5,5)$, $\vec{v} = 4\vec{i} + 3\vec{j}$.
8. Find the tangent plane and normal line for $2z - x^2 = 0$, $P_0(2,0,2)$.
9. Find all local maxima, minima, and saddle points for $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$.
10. Find absolute maxima and minima for $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant.
11. Find the maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.
12. Compute the area bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$ by an iterated integral using (a) vertical cross-sections and (b) horizontal cross-sections.
13. Sketch the region and evaluate $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$.
14. Change to polar and evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy\,dx$.
15. For the region below $z = 4 - xy$ and in $0 \le x \le 2$, $0 \le y \le 1$, integrate the function $3 - 4x$.
16. Use $x = \frac{u}{v}$, $y = uv$ in $\mathbb{R}$ (first quadrant) bounded by $xy = 1$, $xy = 9$, $y = x$, $y = 4x$ for $\displaystyle\iint \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\,dy$.
17. **Evaluate the line integral $\int (xy + y + z)\,ds$ along the curve $r(t) = 2t\vec{i} + t\vec{j} + (2-2t)\vec{k}$, $0 \le t \le 1$.
18. **Find the flow of the field $F = -4xy\vec{i} + y8\vec{j} + 2\vec{k}$ along $r(t) = t\vec{i} + t^2\vec{j} + \vec{k}$, $0 \le t \le 2$.
19. **Find the circulation and flux of the field $F = x\vec{i} + y\vec{j}$ around $r(t) = \cos t\vec{i} + \sin t\vec{j}$, $0 \le t \le 2\pi$.
20. **Find the potential function for $F = e^{y+2z}(\vec{i} + x\vec{j} + 2x\vec{k})$.

21. **Use Green's Theorem to find counterclockwise circulation and outward flux for field $F = (y^2 - x^2)\vec{i} + (x^2 + y^2)\vec{j}$ and curve C: $y = 0$, $x = 3$, $y = x$.

22. **Use a parametrization to express the area of the surface as a double integral: the portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$.

23. Evaluate $\iint 2y\, dV$ over $S : y^2 + z^2 = 4$ between $x = 0$, $x = 3 - z$.

24. At time $t = 0$, a particle is located at the point $(0, 1, 2)$. At this time it is traveling towards the point $(3, 2, 3)$, has speed 3 at $(0, 1, 2)$, and has constant acceleration $3\vec{i} - \vec{j} + \vec{k}$. Find an equation for the position vector $\vec{r}(t)$ at time t.

25. Let $P(A_0, B_0, C_0)$, $Q(A_1, B_1, C_1)$ be two distinct points in 3-dimensional space.
   (a) Write down a vector parameterization for the line PQ.
   (b) Show that the curvature κ of PQ is 0.
   (c) **Show that the torsion τ of PQ is 0.

26. (a) At what points (x, y) in the plane is the function $f(x, y) = \frac{y}{1+\cos x}$ continuous?
   (b) Find the following limit by first rewriting the fraction $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{x^3 + y^3}{x + y}$.

27. Let C be the smooth curve given by the intersection of the two surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$.
   (a) Write down an equation that describes C implicitly.
   (b) Write down a parametric equation for the line tangent to C at the point $(1, 1, 1)$.

28. Let D be the cylinder bounded below by $z = -1$, bounded on the sides by $x^2 + y^2 = 1$, and bounded above by $z = 1$. (**Do not evaluate the integrals.**)
   (a) Express the volume of D as an iterated triple integral in cylindrical coordinates.
   (b) Express the volume of D as an iterated triple integral in rectangular coordinates.
   (c) Express the volume of D as an iterated triple integral in spherical coordinates.

29. Consider the function $f(x, y) = \dfrac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$.
   (a) Describe the region R given by $1 \le x^2 + y^2 \le e$ in polar coordinates.
   (b) Write down the integral of $f(x, y)$ over R in polar coordinates.
   (c) Evaluate the integral you found in part (b).

30. Let D be the region in xyz-space defined by the inequalities $1 \le x \le 2$, $0 \le xy \le 2$, $0 \le z \le 1$. Consider the coordinate transformation of D to the uvw-plane given by $u = x$, $v = xy$, $w = 3z$.
   (a) Sketch the preimage G of D under the coordinate transformation in the uvw-plane and label the bounding curves.
   (b) Write down the Jacobian associated to this coordinate transformation.
   (c) Evaluate the integral $\displaystyle\iiint_D (x^2y + 3xyz)dx\,dy\,dz$.

31. A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. Points on the plate have temperature $T(x, y) = x^2 + 2y^2 - x$. Find the temperatures of the hottest and coldest points of the plate.

32. Find the points on the surface $xyz = 1$ closest to the origin.

33. Find the volume of the wedge cut from the cylinder $x^2 + y^2 \le 1$ by the planes $z = -y$ and $z = 0$.

Answers

1. $T = \frac{3\cos t}{5}\vec{i} - \frac{3\sin t}{5}\vec{j} + \frac{4}{5}\vec{k}$, $N = (-\sin t)\vec{i} - (\cos t)\vec{j}$, $\kappa = \frac{3}{25}$

2. $a(1) = \frac{4}{3}T + \frac{2\sqrt{5}}{3}N$

3. $\frac{5}{2}$

4. $w_{xx} = 6xy\cos(x^2y) - 4x^3y^2\sin(x^2y)$, $w_{yy} = -x^5\sin(x^2y)$, $w_{xy} = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$

5. $\frac{dw}{dt} = 4t\arctan t + 1 = \pi + 1$

6. $\frac{1}{4}, -\frac{3}{4}$

7. -4

8. $2x - z - 2 = 0$, $r(t) = (2, 0, 2) + t(-4, 0, 2)$

9. $f(-3, 3) = -5$ minimum

10. $(0, 0) = 1$, abs max, $(1, 2) = -5$, abs min

11. 39

12. (a) Vertical: $\int_0^9 \int_0^{\sqrt{x}} dydx = 18$; (b) Horizontal: $\int_0^3 \int_{y^2}^9 dxdy = 18$
13. $e - 2$

14. $\frac{\pi}{2}$

15. $\int_0^2 \int_0^1 \int_0^{4-xy} (3 - 4x)dz\, dy\, dx = -\frac{17}{3}$

16. $\int_1^2 \int_1^3 \frac{(u+v)2u}{v} du\, dv = 8 + \frac{52}{3}\ln 2$

17. $\frac{13}{2}$

18. 48

19. Circulation $= 0$; Flux $= 2\pi$

20. $f(x, y, z) = xe^{y+2z} + C$

21. Flux $= -9$, Circulation $= 9$

22. $\int_0^{2\pi} \int_1^3 r\sqrt{5}dr\, d\theta$

23. $r(u, v) = r(x, \theta) = \langle x, 2\sin\theta, 2\cos\theta\rangle$
$$\iint f(r(u, v))|r_u \times r_v|dA = \int_0^{2\pi} \int_0^{3-2\cos\theta} 2(2\sin\theta)(2)dx\, d\theta = 0$$

24. $\vec{r}(t) = \left(\frac{3}{2}t^2 + \frac{9}{\sqrt{11}}t\right)\vec{i} + \left(-\frac{1}{2}t^2 + \frac{3}{\sqrt{11}}t + 1\right)\vec{j} + \left(\frac{1}{2}t^2 + \frac{3}{\sqrt{11}}t + 2\right)\vec{k}$

25. (a)$L(t) = \langle x(t) = A_0 + (A_1 - A_0)t, y(t) = B_0 + (B_1 - B_0)t, z(t) = C_0 + (C_1 - C_0)t\rangle$

26. (a) f is continuous on $\{(x,y)|x \neq (2k+1)\pi; k = \cdots, -1, 0, 1, \cdots\}$. (b) 1

27. (a) $x^2 + 2y^2 + 3z^2 = 6xyz$. (b) $L(t) = \langle 1+2t, 1-4t, 1+2t \rangle$

28. (a) $\int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{1} r\, dr\, d\theta\, dz$. (b) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{-1}^{1} dz\, dx\, dy$
(c) $\int_{0}^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{0}^{1/\sin\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta + 2 \int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{0}^{1/\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

29. (a) $1 \leq r^2 \leq e$. (b) $\int_{0}^{2\pi} \int_{1}^{\sqrt{e}} 2\ln(r) dr\, d\theta$. (c) $-2\pi\sqrt{e} + 4\pi$

30. (b) $\frac{1}{3u}$. (c) $2 + 3\ln(2)$

31. Hottest: $\frac{9}{4}$ at points $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Coldest: $-\frac{1}{4}$ at point $(\frac{1}{2}, 0)$.

32. Points $(1,1,1)$, $(-1,-1,-1)$, $(1,-1,-1)$, and $(-1,1,-1)$ are all at distance 3 from the origin.

33. $\frac{2}{3}$