Reading Day Problem Bank
Chapters 1-4

1. Evaluate the following derivatives; you need not simplify your answers.

   (a) $\frac{d}{dx}\left[x^2\log_3(2x+1)\right]$

   Solution. $\frac{d}{dx}\left[x^2\log_3(2x+1)\right] = \frac{1}{\ln 3}\left(x^2\cdot\frac{1}{2x+1}\cdot 2 + 2x\ln(2x+1)\right)$

   (b) *If* $y = (\tan 2)e^{\sqrt{\cos t}}$, *find* $\frac{dy}{dt}$.

   Solution. Since $\tan 2$ is a constant, $\frac{dy}{dt} = (\tan 2)e^{\sqrt{\cos t}}\cdot\frac{1}{2}(\cos t)^{-1/2}(-\sin t)$

   (c) *If* $g(x) = \sec(x^2e^x)$, *find* $g'(x)$.

   Solution. $g'(x) = \sec\left(x^2e^x\right)\tan\left(x^2e^x\right)\left(x^2e^x + e^x\cdot 2x\right)$

   (d) *If* $f(t) = \dfrac{t}{5\sqrt[3]{1+e^t}}$, *find* $f'(t)$. *(Can you rewrite f so that you don't need to use the Quotient Rule?)*

   Solution.

   $f'(t) = \frac{1}{5}\left(-\frac{1}{3}t\left(1+e^t\right)^{-\frac{4}{3}}e^t + \left(1+e^t\right)^{-\frac{1}{3}}\right)$

   (e) $f(x) = x\sin\left(x^3\right)$, $g(x) = x\sin^3 x$, *and* $h(x) = x(\sin x)^3$. *(Are these the same or different?)*

   Solution.

   $f'(x) = (\sin x)^3 + x\cdot 3(\sin x)^2\cos x$

2. Compute derivatives for the following functions:

   (a) $f(x) = \ln(\sec(\ln x))$.

   Solution.

   $f'(x) = \frac{1}{\sec(\ln x)}\cdot\sec(\ln x)\tan(\ln x)\cdot\frac{1}{x}$

   (b) $f(x) = \arccos(e^{3-x^2})$.

   Solution. $f'(x) = -\frac{1}{\sqrt{1-(e^{3-x^2})^2}}\cdot e^{3-x^2}\cdot(-2x)$

   (c) $f(x) = (\ln x)^{\ln x}$.

   Solution. This is of the form $f(x)^{g(x)}$. So this would need logarithmic differentiation. $f'(x) = (\ln x)^{\ln x}\cdot\left(\frac{1}{x}\cdot\ln(\ln x) + \ln x\cdot\frac{1}{\ln x}\cdot\frac{1}{x}\right)$

   (d) $f(x) = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$.

   Solution.

   $f'(x) = \boxed{\sqrt[3]{\frac{x(x-2)}{x^2+1}}\cdot\left[\frac{1}{3}\left(\frac{1}{x}+\frac{1}{x-2}-\frac{2x}{x^2+1}\right)\right]}$ Note: if you use a different method for computing the derivative, it is possible that you get an answer that has a completely different form and it is non-trivial to verify that the two answers are actually the same. If you are interested, you can check out the answer Wolframalpha gives. That answer is actually the same as this one.

(e) $u(x) = \sin\left(\tan^{-1}(\ln x)\right)$

Solution. $u'(x) = \cos\left(\tan^{-1}(\ln x)\right) \cdot \frac{1}{1+(\ln x)^2} \cdot \frac{1}{x}$

(f) $a(t) = \dfrac{(t+2)^3(t+5)^7}{\sqrt{t-5}}$

Solution. $a'(t) = \frac{(t+2)^3(t+5)^7}{\sqrt{t-5}} \cdot \left(\frac{3}{t+2} + \frac{7}{t+5} - \frac{1}{2(t-5)}\right)$

(g) $s(t) = \log_2\left(\log_3\left(e^{t^2}\right)\right)$

Solution. $s'(t) = \frac{1}{\log_3\left(e^{t^2}\right)\ln 2} \cdot \frac{1}{\log_3(e^{t^2})} \cdot e^{t^2} \cdot 2t$

(h) $h(x) = \dfrac{\sqrt{\arctan(4x)}}{e^5}$

Solution. $h'(x) = \boxed{\dfrac{1}{e^5} \cdot \dfrac{1}{2} \cdot \arctan(4x)^{-1/2} \cdot \left(\dfrac{1}{1+(4x)^2} \cdot 4\right)}$

(i) $f(t) = 3^t + t^3 + (\ln t)^{3t}$

Solution. $f'(t) = \boxed{3^t \ln 3 + 3t^2 + (\ln t)^{3t}\left[\dfrac{3}{\ln t} + 3\ln(\ln t)\right]}.$

(j) $f(x) = \log_3(\arctan(x)\arcsin(x))$.

Solution. This needs the chain rule, and then the product rule for the derivative of the inside function:

$$\begin{aligned} f'(x) &= \frac{1}{\arctan(x)\arcsin(x)} \cdot \frac{1}{\ln(3)} \cdot \left[\frac{1}{1+x^2} \cdot \arcsin(x) + \arctan(x) \cdot \frac{1}{\sqrt{1-x^2}}\right] \\ &= \frac{1}{\ln(3)\arctan(x)(1+x^2)} + \frac{1}{\ln(3)\arcsin(x)\sqrt{1-x^2}} \end{aligned}$$

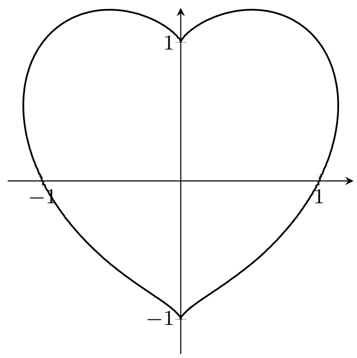
3. *Find the equation of the tangent line to the curve*

$$e^{2x} = \sin(x^2 + 2y) + 1.$$

at the point (0,0).

Solution. The line has slope 1. Since $(0,0)$ is a point on the line, the equation of the line is $\boxed{y = x}$.

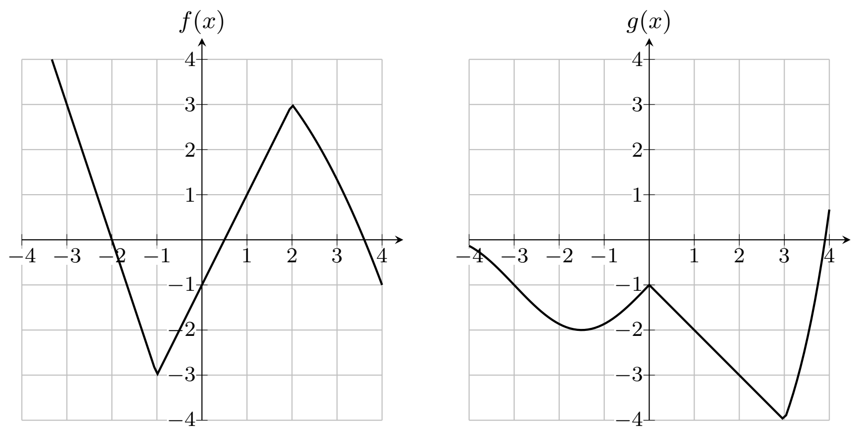
4. *Find the line tangent to the heart curve* $(x^2 + y^2 - 1)^3 - x^2y^3 = 0$ *at* $(-1, 1)$.



Solution.

$\frac{dy}{dx} = \frac{4}{3}$, so the equation of the line is $\boxed{y - 1 = \frac{4}{3}(x+1)}$. (From the picture, we can see that the tangent line should indeed have positive slope.)

5. *Here are graphs of two functions, $f(x)$ and $g(x)$. If $F(x) = f(g(x))$, what is $F'(1)$?*



Solution. By the Chain Rule, $F'(1) = f'(g(1))g'(1)$. From the graph of g, we see that $g(1) = -2$ and $g'(1) = -1$. Therefore, $F'(1) = -f'(-2)$. From the graph of f, we see that $f'(-2) = -3$, so $F'(1)$ is equal to $\boxed{3}$.

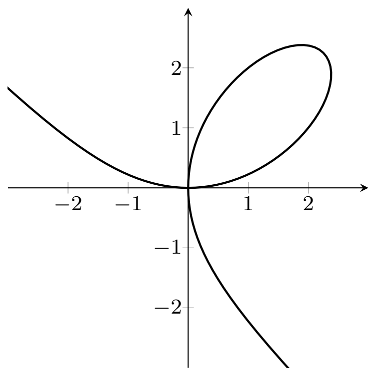
6. *You are given the following information about three functions f, g, and h.*

$$h(1) = 2 \qquad g(2) = 3 \qquad f'(3) = 6$$
$$h'(1) = 4 \qquad g'(2) = 5$$

If $r(x) = f(g(h(x)))$, do you have enough information to find $r'(1)$? If so, compute it. If not, what additional information do you need?

Solution. $r'(1) = \boxed{120}$

7. *Look again at the folium of Descartes, $x^3 + y^3 = \frac{9}{2}xy$. Find all points on the curve where the tangent line is horizontal.*



Solution. The only point on the curve with a horizontal tangent line is $\boxed{\left(\frac{3}{\sqrt[3]{4}}, \frac{3}{\sqrt[3]{2}}\right)}$.

8. Suppose you want to compute derivatives of the following functions. For which would you want to use logarithmic differentiation? Which could you do without logarithmic differentiation? You do not need to actually compute these derivatives!

   (a) $(x^3 + 17\sin x)^{\ln 3}$

   Solution. We don't need log differentiation here. This function is of the form $[f(x)]^b$ so to compute the derivative, we can use the chain rule!

$$\frac{d}{dx}\left(\boxed{(\boxed{x^3 + 17\sin x})^{\ln 3}}\right) = \ln 3(x^3 + 17\sin x)^{\ln 3 - 1} \cdot (3x^2 + 17\cos x)$$

   (b) $7^{\sqrt{x^3+14}}$

   Solution. We don't need log differentiation here. This function is of the form $[b]^{f(x)}$ so to compute the derivative, we can use the chain rule!

$$\frac{d}{dx}\left(\boxed{7^{\boxed{\sqrt{x^3+14}}}}\right) = 7^{\sqrt{x^3+14}} \ln 7 \cdot \frac{d}{dx}\left(\boxed{\sqrt{\boxed{x^3 + 14}}}\right)$$
$$= 7^{\sqrt{x^3+14}} \ln 7 \cdot \frac{1}{2}(x^3 + 14)^{-1/2} \cdot (3x^2)$$

   (c) $(x^2 + \sqrt{x})^{\tan x}$

   Solution. Here logarithmic differentiation is *essential* because this function is of the form $f(x)^{g(x)}$. Let $y = (x^2 + \sqrt{x})^{\tan x}$ We will start by taking logs on both side, and then differentiate:

$$\ln y = \ln\left[(x^2 + \sqrt{x})^{\tan x}\right]$$
$$\ln y = \tan x \cdot \ln(x^2 + \sqrt{x})$$

   Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(x^2 + \sqrt{x}) + \tan x \cdot \frac{1}{x^2 + \sqrt{x}} \cdot \left(2x + \frac{1}{2}x^{-1/2}\right)$$

9. *Two cars are approaching an intersection. A red car, approaching from the north, is traveling 20 feet per second and is currently 60 feet from the intersection. A blue car, approaching from the west, is traveling 30 feet per second and is currently 80 feet from the intersection. At this moment, is the distance between the two cars increasing or decreasing? How quickly?*

   Solution. The distance between the cars is $\boxed{\text{decreasing at an instantaneous rate of 36 ft/s}}$.

10. *An oil tank in the shape of an inverted cone has height 10 m and radius 6 m. When the oil is 5 m deep, the tank is leaking oil from the tip at a rate of 2 m^3 per day. How quickly is the height of the oil in the tank decreasing at this moment?*

    Note: The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.

    Solution. So, at the time we're interested in, the height of oil is $\boxed{\text{decreasing at } \frac{2}{9\pi} \text{ m/day}}$

11. *At noon, you are running to get to class and notice a friend 100 feet west of you, also running to class. If you are running south at a constant rate of 450 ft/min (approximately 5 mph) and your friend is running north at a constant rate of 350 ft/min (approximately 4 mph), how fast is the distance between you and your friend changing at 12:02 pm?*

    Solution. The distance between you and your friend is $\boxed{\text{increasing at } \dfrac{12800}{\sqrt{257}} \text{ ft/min}}$ at 12:02.

12. During a night run, an observer is standing 80 feet away from a long, straight fence when she notices a runner running along it, getting closer to her. She points her flashlight at him and keeps it on him as he runs.

    When the distance between her and the runner is 100 feet, he is running at 9 feet per second. At this moment, at what rate is she turning the flashlight to keep him illuminated? Include units in your answer.

    Solution.

    The observer must turn the flashlight at $\boxed{\dfrac{9}{125} \text{ radians per second}}$.

13. As you're riding up an elevator, you spot a duck on the ground, waddling straight towards the base of the elevator. The elevator is rising at a speed of 10 feet per second, and the duck is moving at 5 feet per second towards the base of the elevator. As you pass the eighth floor, 100 feet up from the level of the river, the duck is 200 feet away from the base of the elevator. At this instant, at what rate is the distance between you and the duck changing?

    Solution. $\frac{dz}{dt} = \boxed{0 \text{ ft/s}}$.

14. *Kelly is flying a kite; the kite is 100 ft above the ground and moving horizontally away from Kelly. At precisely 1 pm, Kelly has let out 300 ft of string, and the amount of string let out is increasing at a rate of 5 ft/s. If Kelly is standing still, at what rate is the angle between the string and the vertical increasing at 1 pm? (You may assume that the string is stretched taut so that it is a straight line.)*

    Solution.

    $\frac{d\theta}{dt} = \boxed{\dfrac{1}{120\sqrt{2}} \text{ radians per second}}$

15. *A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at a rate of 5 ft/sec.*

   (a) *How fast is the top of the ladder sliding down the wall then?*

   (b) *At what rate is the area of the triangle formed by the ladder, wall and ground changing then?*

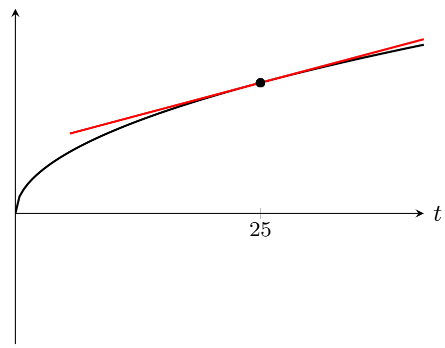
   (c) *At what rate is the angle between the ladder and the ground changing then?*

Solution. $\frac{d\theta}{dt} = \frac{12^2}{13^2}\frac{-12*12-5*5}{12^2} = \frac{-169}{13^2} = -1\frac{rad}{sec}$.

16. (a) *Use linear approximation to estimate $\sqrt{24.5}$ without using a calculator. Draw a sketch to explain what you are doing.*

   Solution. $\sqrt{24.5} \approx 5 + \frac{1}{10}(-0.5) = 4.95$.

   Here's a picture of our situation:



   (b) *From your sketch, you should be able to tell whether your approximation is an overestimate or underestimate. Which is it?*

   Solution. From the sketch it is clear that our estimate is an overestimate.

   (c) *Explain your answer to (b) using a second derivative.*

   Solution. $f''(x) = \frac{-1}{4}x^{-3/2} = \frac{-1}{4} \cdot \frac{1}{\sqrt{x^3}}$. Since this is negative for any value of x, the function is concave down, and so the estimate is an over-estimate.

17. *Use linear approximation to estimate $9^{4/3}$. Is your estimate too high or too low?*

Solution. $9^{4/3} \approx 16 + \frac{8}{3} = \boxed{18 + \frac{2}{3}}$. The second derivative $f''(x)$ is $\frac{4}{9}x^{-2/3}$, which is positive on $(0, \infty)$; therefore, f is concave up on $(0, \infty)$. So, our tangent line approximation must have been under the graph of f, which means that our estimate was $\boxed{\text{too low}}$.

18. In this problem, we'll look at the cubic function $f(x) = x^3 + 3x^2 + 1$.

   (a) *Find all critical points of f.*

   Solution. $f'(x) = 3x^2 + 6x = 3(x^2 + 2x) = 3x(x + 2)$, so $f'(x) = 0$ at $x = \boxed{-2, 0}$. (There are no points where f' is undefined.)

(b) *Make a sign chart for f', and use this to decide whether each of the critical points you found is a local minimum, a local maximum, or neither.*

Solution. At $x = -2$ there is a local max and at $x = 0$ there is a local minimum.

19. Let $f(x) = x^4 - 8x^2 + 16$.

(a) *Find all critical points of f.*

Solution. The critical points are

$$x = -2,\ 0, \text{and } 2$$

(b) *For each critical point c of f, find the sign of $f''(c)$. What does this tell you about the critical point c?*

Solution. $f''(x) = 12x^2 - 16$

x	$f''(x)$
0	-16
2	32
-2	32

20. Suppose you've made the following sign chart for the derivative of a function $f(x)$. The function $f(x)$ is continuous and differentiable on $(-\infty, \infty)$.

		0		3		5	
sign of f'	$+$		$-$		$+$		$-$

(a) *Does $f(x)$ have an absolute maximum on $(-\infty, \infty)$? (Definitely yes, definitely no, or maybe?) If so, where could it be?*

Solution. The function definitely has an absolute maximum at either $x = 0$ or at $x = 5$.

(b) *Does $f(x)$ have an absolute minimum on $(-\infty, \infty)$? If so, where could it be?*

Solution. There is not enough information to be sure.

(c) *Does $f(x)$ have an absolute maximum on $[-2, 10]$? If so, where could it be?*

Solution. Yes, f must have an absolute maximum on $[-2, 10]$; it could be at $x = 0$ or $x = 5$.

(d) *Does $f(x)$ have an absolute minimum on $[-2, 10]$? If so, where could it be?*

Solution. Yes, f must have an absolute minimum on $[-2, 10]$; it could be at $x = -2$, $x = 3$, or $x = 10$.

21. *Let $f(x) = 3x^{1/3} + 4x$. Find all critical points of f, and determine whether each critical point is a local minimum, local maximum, or neither.*

Solution. The critical point $x = 0$ is neither a local minimum nor a local maximum.

22. *Sanity Check. Do this without looking at your notes! Which of the following is the linearization of a function $f(x)$ at $x = a$?*

A. $y = f(a) + f'(x)(x - a)$

B. $y = f(x) + f'(a)(x - a)$

C. $y = f(x) + f'(x)(x - a)$

(D.) $y = f(a) + f'(a)(x - a)$

E. $y = f'(a) + f(x)(x - a)$

F. $y = f'(x) + f(a)(x - a)$

G. $y = f'(x) + f(x)(x - a)$

H. $y = f'(a) + f(a)(x - a)$

23. (a) *Use linear approximation to approximate the value of* $\cos\left(\frac{93}{180}\pi\right)$ *and* $\cos\left(\frac{86}{180}\pi\right)$.

Solution. $\cos\left(\frac{93}{180}\pi\right) \approx -\frac{93}{180}\pi + \frac{\pi}{2} = -\frac{3}{180}\pi = \frac{-\pi}{60}$

$\cos\left(\frac{86}{180}\pi\right) \approx -\frac{86}{180}\pi + \frac{\pi}{2} = \frac{4}{180}\pi = \frac{\pi}{45}$

(b) *To determine if the estimates are over or under the actual values, it is helpful to determine the concavity of the function. On the interval* $[0,\pi]$ *where is* $f(x) = \cos(x)$ *concave up? On the interval* $[0,\pi]$ *where is* $\cos(x)$ *concave down? Justify your answer using calculus.*

Solution.

- $f''(x)$ is negative on the interval $(0, \frac{\pi}{2})$ and so $f(x)$ is concave down.
- $f''(x)$ is positive on the interval $(\frac{\pi}{2}, \pi)$ and so $f(x)$ is concave up.

(c) *Are your estimates overestimates or underestimates? Sketch a graph to justify.*

Solution. $\cos\left(\frac{86}{180}\pi\right) \approx \frac{4}{180}\pi$ is an overestimate while $\cos\left(\frac{93}{180}\pi\right) \approx -\frac{3}{180}\pi$ is an underestimate.

24. *Use linear approximation to estimate* $e^{0.1}$. *Explain whether your estimate is overestimate of under estimate.*

Solution.

$$e^{0.1} = f(0.1) \approx 0.1 + 1 = 1.1.$$

To find out whether this estimate is over or under estimate, we compute the second derivative and see that

$$f''(x) = e^x,$$

This is always positive, and so $f(x)$ is concave up. That is, the estimate is an underestimate.

25. *Use linear approximation to estimate* $\sin^2(\frac{\pi}{4} - 0.1)$.

Solution. $f(\frac{\pi}{4} - 0.1) \approx \frac{1}{2} + (\frac{\pi}{4} - 0.1 - \frac{\pi}{4}) = 0.4$.

26. Consider $f(x) = x - 2 + \frac{1}{x}$. Find all local and extrema of this function on the following intervals:

(a) $(0,\infty)$

Solution. There is a local and absolute minimum at $x = 1$. There is no global maximum or local maximum.

(b) [0.5,5]

Solution. The local and global minimum is at $x = 1$. Also, there are local maximums at both $x = 0.5$ and $x = 5$. In order to work out global maximums, we need to figure out what is happening at the ends of our domain. There is a global maximum at either $x = 0.5$ or at $x = 5$. We decide by just plugging both points in for x and computing. Since $f(0.5) = 0.5$, and $f(5) = 3.2$, we see that f has a global max at $x = 5$.

(c) (0,1)

Solution. Now there are no critical points or end points to consider on this domain. From the sign chart, we can see that the function is decreasing on (0,1). But since the end points are not included, there are no local or global extrema.

27. Find all critical points of the given functions, and determine whether each critical point is a local minimum, local maximum, or neither. Be flexible; you don't need to use the same strategy for each part!

(a) $f(x) = 8x - e^x$

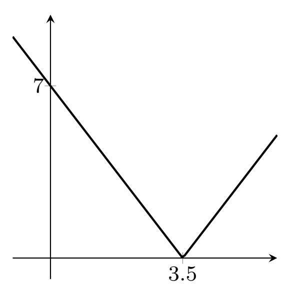
Solution. $f'(x) = 8 - e^x$, so the only critical point is $x = \ln 8$. It looks like f'' is simple, so let's use the Second Derivative Test to classify this critical point: $f''(x) = -e^x$, which is always negative, so $f''(\ln 8) < 0$, and f has a $\boxed{\text{local maximum at } \ln 8}$.

(b) $f(t) = t^4 + t^3$

Solution. $f'(t) = 4t^3 + 3t^2 = t^2(4t + 3)$ so the critical points are $t = 0, -\frac{3}{4}$. The the critical point at $x = 0$ is neither a local minimum nor local maximum, while the critical point at $x = -\frac{3}{4}$ is a local minimum.

(c) $g(t) = |2t - 7|$

Solution. We don't have a nice formula for differentiating absolute values, but this function is easy to graph: the graph of $2t - 7$ is a line with y-intercept -7 and t-intercept 3.5, so here is the graph of g:



From the graph, we see that $\boxed{\text{the only critical point is 3.5, and } g \text{ has a local minimum there}}$.

28. (a) *Does* $f(r) = 2\pi r^2 + \frac{256\pi}{r}$ *have an absolute maximum and absolute minimum on* $[1, 8]$*? If so, where? (Give the* r*-values at which the absolute minimum and absolute maximum occur.)*

Solution. $(1{,}258\pi)$ is the absolute max and $(4, 96\pi)$ is the absolute min.

(b) *Does* f *have an absolute minimum and absolute maximum on* $(0, \infty)$*? If so, where?*

Solution. The Extreme Value Theorem no longer applies since our function is not defined on a closed interval. The only critical point is at 4, and there are no end points to consider. From our analysis above we see 4 as a absolute minimum.

To check whether there is a maximum or not, we need to consider what is happening at the ends of our domain. Since the domain is $(0,\infty)$, we do this by computing the limits $\lim_{r\to\infty} f(r)$ and $\lim_{r\to 0^+} f(r)$. If either one of them is ∞, we know that there is no absolute maximum! Indeed here since

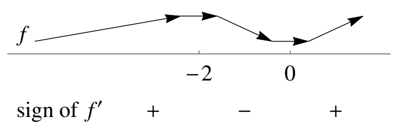
$$\lim_{r\to\infty} f(r) = \infty$$

this function does not obtain an absolute max.

29. *Last time, we looked at $f(x) = x^3 + 3x^2 + 1$ and found that it had the following local extrema:*

- *A local maximum at $x = -2$*
- *A local minimum at $x = 0$*

Here is the sign chart again for this function:



Does f have an absolute maximum and absolute minimum on $(-3,3)$?

Solution. We need to compare $f(-2)$ with $\lim_{x\to 3^-} f(x)$ to decide if the function has an absolute max.

$$f(-2) = 5 \text{ and } \lim_{x\to 3^-} f(x) = 55$$

This shows that the function does not obtain an absolute maximum. To decide if the function has an absolute min or not we need to compare $f(0)$ with $\lim_{x\to -3^+} f(x)$

$$f(0) = 1 \text{ and } \lim_{x\to -3^+} f(x) = 1$$

So the function does have an absolute min at $x = 0$.

30. Let

$$f(x) = \begin{cases} -x^2 - 4x + 4 & x \le 1, \\ -x^2 + 5x - 5 & x > 1 \end{cases}$$

(a) *Find all the critical points of $f(x)$.*

Solution. The critical points of $f(x)$ are $x = 1, -2, 5/2$.

(b) *Identify each critical point as a local maxima or local minima of $f(x)$.*

Solution. The local maxima are $(-2, 8)$ and $(5/2, 5/4)$. Local minima is $(1, -1)$.

(c) *Find the absolution maximum and minimum of $f(x)$ on $[-3,3]$.*

Solution. The absolute maximum is $f(-2) = 8$ and absolute minimum is $f(1) = -1$.

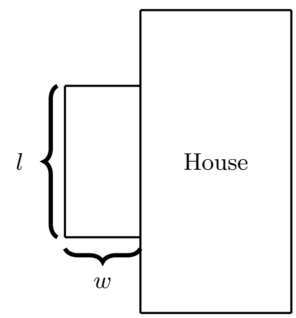
31. Sarah has a new puppy and she wants to maximize her outdoor time, so she builds her a fenced in play area. She has 40 feet of fencing, and she wants to fence off a rectangular area next to her house. The house will be one side of the play area, so that side needs no fencing. In order for the puppy to have adequate space, the area needs to be at least 5 feet long and 5 feet wide. What is the largest area she could have? Is there a smallest area she could have if she wants to use all 40 feet of fencing?

   (a) *What's the goal? (That is, what are we trying to maximize or minimize?)*

   Solution. We want to maximize area of the play pen. It is super important to articulate the goal of the problem. It sets the tone for everything to follow. A lot of times students mess up at this step, and that derails the whole problem! So please take a breath, and think through this carefully before proceeding on any optimization problem.

   (b) *To help make sense of the problem. Draw a picture. Introduce some useful variables.*

   Solution. Here's a picture and some variables that might be useful!



   (c) *Express the thing we're trying to maximize or minimize as a function of one variable. What's the domain of this function?*

   Solution. Once we have a goal that is clearly articulated and we have a understanding of what the problem is asking, we can formulate a strategy. Because we want to optimize area we should try and find a single variable function for the area of the play area. Once we have a single variable function we can use the Closed Interval Test.

$$A = wl$$

   Note that we start off with area the function of two variables, but in order to make use of the calculus we have learned we need a function of one variable. We have 40 feet of fencing available so there is a relationship between the variables

$$40 = 2w + l$$

$$40 - 2w = l$$

   So we can find area as a function of w,

$$A(w) = w(40 - 2w)$$

   What is the domain of this function? From the problem statement we know that $5 \leq w$. Since there is only so much fencing, there is an upper bound to w too. That upper bound happens when l is at its lower bound:

$$40 = 2w + 5$$

$$w = 17.5$$

   So, our new goal is to optimize $A(w) = w(40 - 2w)$ on the domain [5,17.5]

(d) *Finish the problem by actually maximizing / minimizing the function you found in (c) on its domain.*

Solution. The critical points are the points where endpoints $A'(w) = 40 - 4w = 0$ or $w = 10$. There are no points where $A'(w)$ does not exist. We also need to consider the end points $w = 17.5$ and $w = 5$. To determine the maximum area we can now use the Closed Interval Test.

w	5	10	17.5
$A(w)$	150	200	87.5

This shows the maximum area of 200 ft^2 occurs when $w = 10$. We can also see that the minimum area of 87.5 ft^2 occurs when $w = 17.5$.

(e) *Finally, go back and make sure you actually answered the question!*

Solution. Done and dusted!

32. *Walter is planning to buy a custom-made jewelry box for his mother's next birthday. The box will have a square base. The sides and bottom will be made out of mahogany, which costs 30 cents per square inch. The top will be made out of maple, which costs 50 cents per square inch. Walter has $60 to spend on the present and wants to get a box with the largest volume possible. What dimensions should his box be?*

Solution. The dimensions for the box should be $\boxed{5 \text{ inches} \times 5 \text{ inches} \times \frac{20}{3} \text{ inches}}$.

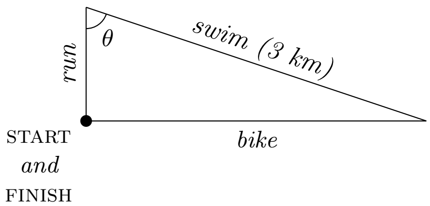
33. *Piedmont Park is renovating the grounds. They want to include a beautiful fountain surrounded by a walking path. The fountain will be in the shape of a rectangle. There will be a walking path in the park to enjoy the fountain. The path will follow two opposite sides of the fountain and two semi-circles on the other sides the rectangle. If the track around the fountain is going to be 440 yards long, what dimensions would maximize the area the fountain?*

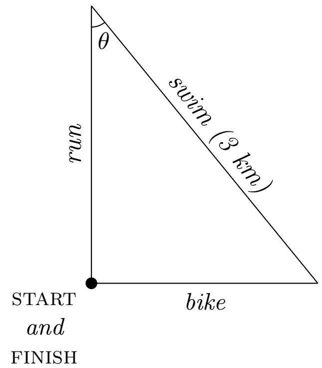
Solution. The track should have $\boxed{\text{semicircles of radius } \frac{110}{\pi} \text{ yards}}$ and $\boxed{\text{straight sides of length 110 yards}}$.

34. *Mel is designing a triathlon course, which will have three legs in the following order: biking, swimming, and running. She decides to make a course in the shape of a right triangle, with the hypotenuse as a 3 kilometer swim. Two possibilities are shown below.*

Mel knows the average triathlete swims at 2 km/hr, runs at 12 km/hr and bikes at $12\sqrt{3}$ *km/hr. In order to make her triathlon a real challenge, she wants to* **maximize** *the time it takes for an average triathlete to complete the course.*

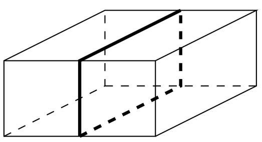
Find the ideal triathlon course for Mel by saying how long the biking and running legs should be.





Solution. The biking leg should be 1.5 km, and the running leg should be $\frac{3\sqrt{3}}{2}$ km.

35. *For the holidays, Paul plans to give his friends gift boxes of his homemade caramel popcorn. He would like to send each friend a box containing 240 cubic inches of popcorn, and he is trying to minimize the cost of each box. Each box will be 6 inches tall and made out of holiday-themed paperboard costing 5 cents per square inch. In addition, Paul plans to wrap a bright red ribbon around each box as shown (the ribbon is the dark black line in the picture). If the ribbon costs 45 cents per inch, what dimensions should the box be to minimize Paul's costs?*



Solution. Paul should make a box with $\boxed{w = 10 \text{ and } \ell = 4}$.

36. *Traditional Chinese fans are often shaped like circular sectors.*[1] *Marvin wants to buy a large decorative fan of perimeter 60 cm to hang on his wall, so he commissions a fan designer to design such a fan for him. Marvin will pay 10 cents per square cm for the fan's surface, and he would like the top curved edge of the fan is to be lined with gold thread costing 15 cents per cm. If the fan designer would like to maximize her revenue, what dimensions should she make the fan?*

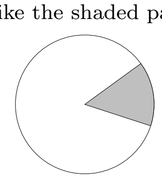
Solution. The designer should make a fan with a $\boxed{\text{radius of } \frac{27}{2} \text{ cm and an inner angle of } \frac{22}{9} \text{ radians}}$.

37. We'd like to get an accurate graph of $f(x) = \dfrac{\pi x^2 + 2}{x}$. Answer the questions below in whatever order makes most sense to you, and use them to sketch the graph of f.

    (a) *What is the domain of f? Does f have any useful symmetry?*

    Solution. The domain of f is all real numbers except $x = 0$. That is the domain is $\boxed{(-\infty,0) \cup (0,\infty)}$. Also, f is $\boxed{\text{odd}}$. This is great! In order to graph f, we can really analyze it on just $(0, \infty)$ and

[1] A sector is like a "pie slice" out of a circular pie, like the shaded part of the circle below:



then use symmetry to fill in the other half of the graph.

(b) *Find and classify all discontinuities of f. Justify your classifications using limits.*

Solution. There is a vertical asymptote at $x = 0$.

(c) *Find all horizontal asymptotes of f.*

Solution. There are no horizontal asymptotes.

(d) *On what intervals is $f(x)$ increasing? On what intervals is it decreasing?*

Solution. f is increasing on $\left(-\infty, -\sqrt{\frac{2}{\pi}}\right) \cup \left(\sqrt{\frac{2}{\pi}}, \infty\right)$ and decreasing on $\left(-\sqrt{\frac{2}{\pi}}, 0\right) \cup \left(0, \sqrt{\frac{2}{\pi}}\right)$

(e) *On what intervals is $f(x)$ concave up? On what intervals is it concave down?*

Solution. f is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$.

(f) *What are the inflection points of f?*

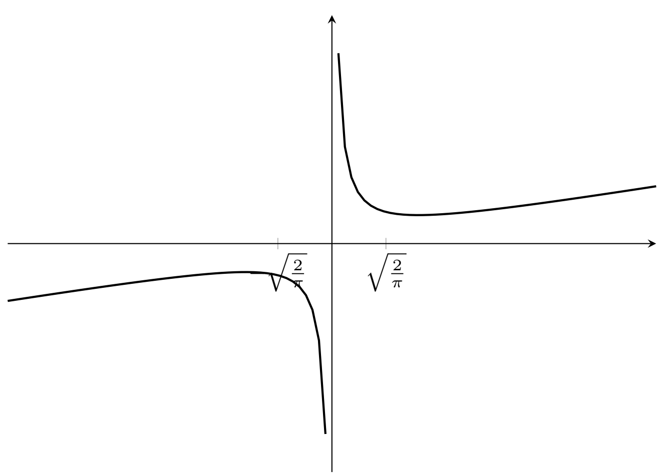
Solution. Although concavity changes at $x = 0$, there isn't an inflection point there because f is undefined at 0.

(g) *Can you find the x- and y-intercepts of f?*

Solution. We found that there are no x and y intercepts. The graph never crosses the axes.

(h) *Sketch a rough graph of $f(x)$ that incorporates all of the above information.*

Solution.



38. Let $f(x) = 3x^{5/3}(x - 8)$.

Answer the questions below in whatever order makes most sense to you, and use them to sketch the graph of f.

(a) *What is the domain of f? Does f have any useful symmetry?*

Solution. $\boxed{\text{The domain of } f \text{ is all real numbers.}}$ Also, f is $\boxed{\text{neither even nor odd}}$.

(b) *Find and classify all discontinuities of f. Justify your classifications using limits.*

Solution. Since f is made up of familiar functions, its discontinuities are exactly where it is undefined. However, we've already said that f is defined everywhere, so f has $\boxed{\text{no discontinuities}}$.

(c) *Find all horizontal asymptotes of f.*

Solution. $\boxed{\lim_{x\to\infty} 3x^{5/3}(x-8) = \infty}$.

As $x \to -\infty$, both $x^{-5/3}$ and $x - 8$ decrease without bound, so $\boxed{\lim_{x\to\infty} 3x^{5/3}(x-8) = \infty}$.

(d) *On what intervals is $f(x)$ increasing? On what intervals is it decreasing?*

Solution. $\boxed{f \text{ is decreasing on } (-\infty, 5) \text{ and increasing on } (5, \infty)}$. (It's also okay to say that f is decreasing on $(-\infty, 0)$ and $(0, 5)$.)

(e) *On what intervals is $f(x)$ concave up? On what intervals is it concave down?*

Solution. $\boxed{f \text{ is concave down on } (0, 2) \text{ and concave up everywhere else}}$.

(f) *What are the inflection points of f?*

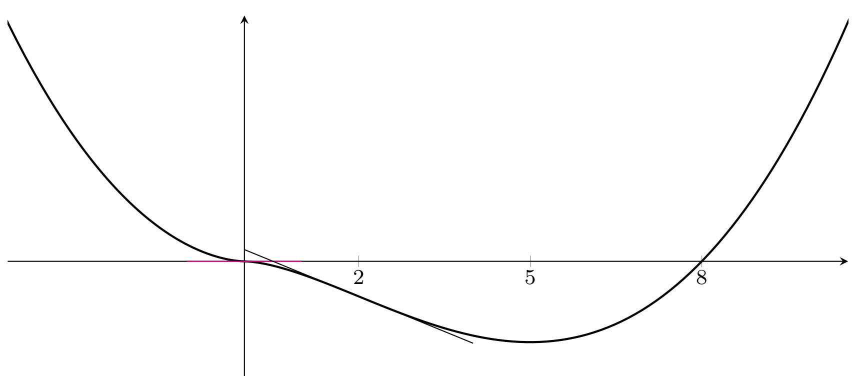
Solution. The concavity of $f(x)$ changes at $x = 0$ and $x = 2$, so there are two inflection points, $\boxed{(0,0) \text{ and } \left(2, -18 \cdot 2^{5/3}\right)}$.

(g) *Can you find the x- and y-intercepts of f?*

Solution. The y-intercept is $f(0) = 0$. The x-intercepts are where $f(x) = 0$, which happens when $x = 0$ or $x = 8$.

(h) *Sketch a rough graph of $f(x)$ that incorporates all of the above information.*

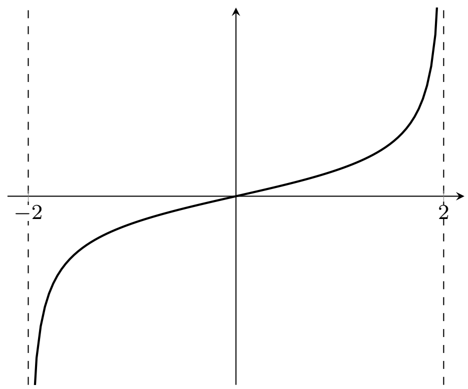
Solution.



(We've drawn the tangent lines at the inflection points to make the concavity change there clearer.)

39. *Graph* $f(x) = \dfrac{x}{\sqrt{4-x^2}}$. *Show how you found the domain, the intervals on which the function is increasing and decreasing, the intervals on which the function is concave up and concave down, the local maxima and minima, any asymptotes, and anything else of interest on the curve.*

Solution.



40. Let $f(x) = \arcsin(4x^2)$.

(a) *What is the domain of* $f(x)$*?*

Solution. $\arcsin u$ is defined for u in $[-1, 1]$, which we can think of as $|u| \le 1$. So, $\arcsin(4x^2)$ is defined on $\boxed{\left[-\frac{1}{2}, \frac{1}{2}\right]}$.

(b) *Is* $f(x)$ *an even function, an odd function, or neither?*

Solution. f is $\boxed{\text{even}}$.

(c) *On what intervals is* f *increasing? On what intervals is it decreasing?*

Solution. $\boxed{f \text{ is decreasing on } \left(-\frac{1}{2}, 0\right) \text{ and increasing on } \left(0, \frac{1}{2}\right).}$

(d) *Does* f *have an absolute maximum and absolute minimum on its domain? If so, find the absolute maximum and minimum values, and say where they occur.*

Solution. The absolute minimum point is $(0, 0)$, and the absolute maximum points are $\left(-\frac{1}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

(e) *In (a), you should have found that the domain of* f *was a closed interval* $[a, b]$*. What are* $\lim_{x\to a^+} f'(x)$ *and* $\lim_{x\to b^-} f'(x)$*? (Note that this is asking about* f'*, not about* f*.) What does this tell you about the graph of* f*?*

Solution. We are asked to calculate $\displaystyle\lim_{x\to(-1/2)^+} \frac{8x}{\sqrt{1-16x^4}}$ and $\displaystyle\lim_{x\to(1/2)^-} \frac{8x}{\sqrt{1-16x^4}}$.

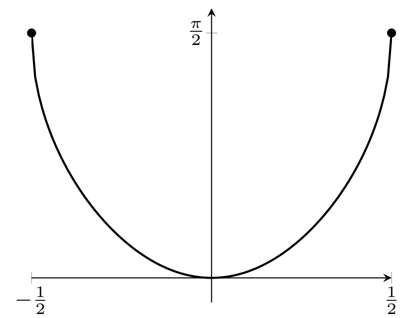
As $x \to \left(-\frac{1}{2}\right)^+$, $8x \to -4$ and $\sqrt{1-16x^4} \to 0$; when we're looking at the limit of a fraction where the numerator has a non-zero limit and the denominator $\to 0$, we need to think about the sign of the denominator. Here, the denominator $\sqrt{1-16x^4}$ is always positive, so $\boxed{\lim_{x\to(-1/2)^+} f'(x) = -\infty}$.

(When x is just a bit bigger than $-\frac{1}{2}$, evaluating $\dfrac{8x}{\sqrt{1-16x^4}}$ involves dividing a number close to -4 by a tiny positive number, which results in a very negative answer.)

As $x \to \left(\frac{1}{2}\right)^-$, $8x \to 4$ and $\sqrt{1-16x^4} \to 0$. Since $\sqrt{1-16x^4}$ is always positive, $\boxed{\lim_{x\to(1/2)^-} f'(x) = \infty}$.

(f) *Use the above information to sketch a rough graph of f. (Your sketch need not accurately reflect the concavity of the graph.)*

Solution. Here is the graph of $f(x)$:



41. *Determine how many roots $f(x) = x^3 + x - 2$ has, if any, on the interval x in* [0,3].

Solution. We've tackled something similar in spirit in the past. Revisit the handout and homework on the Intermediate Value Theorem!

But essentially, before we start counting, let's think about whether there is a root at all on this interval.

Note that the function f is continuous on [0,3] and differentiable on (0,3). Since $f(0) = -2$ and $f(3) = 28$, by the intermediate value theorem, we know that the function takes all values between -2 and 28, and so specifically, it must hit the value 0.

So there is at least one root between [0,3]. Are there more? Suppose there were two points a,b where $f(a) = f(b) = 0$. Then, since this function is continuous and differentiable everywhere, by the Mean Value Theorem, there must be a point c in (a,b) such that:

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

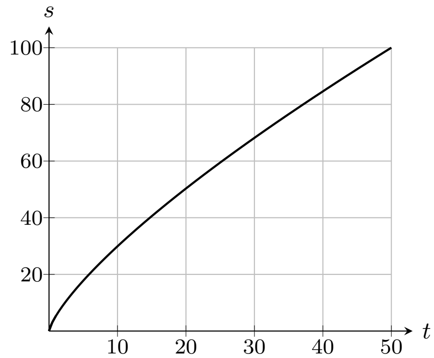
But $\frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$. So we are saying that there must be a point where $f'(c) = 0$. But $f'(x) = 3x^2 + 1$ and this is never equal to zero. Hence, f must have exactly one root in the interval [0,3].

42. Show that $f(x) = x + \ln(x)$ has a zero.

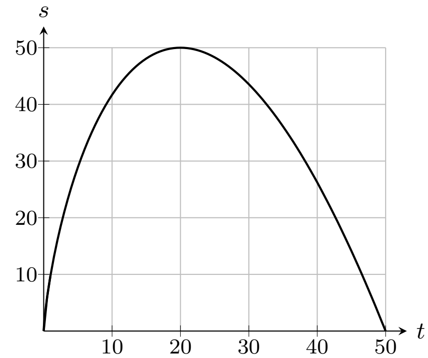
Solution. $\lim_{x\to 0^+} f(x) = -\infty$, and $f(1) = 1 + 0 = 1$. By the IVT, f takes all values in $(-\infty,1)$, including 0.

43. *Average speed vs. average velocity.* A swimmer is swimming a 100 m long race, which is one lap in a 50 m long pool. Let $s(t)$ be his distance from the starting position t seconds after the start of the race.

(a) *Which of the following is a more reasonable graph for $s(t)$? Why?*



(A)



(B)

Solution. Choice (B) is the reasonable one: when the swimmer reaches the opposite end of the pool, he is 50 m from the starting point. At the end of the race, he is back at the starting point, so he is 0 m from the starting point. (Choice (A) would say that, at the end of the race, the swimmer is 100 m from his starting point.)

(b) *According to the graph you chose, what was the swimmer's average speed for the race? Average velocity for the race?*

Solution. First, remember that average speed $= \frac{\text{change in distance traveled}}{\text{change in time}}$, while average velocity $= \frac{\text{change in position}}{\text{change in time}}$.

When the swimmer starts the race, he has traveled 0 m; when the swimmer ends, he has traveled 100 m; therefore, the change in his distance traveled is 100 m. The race takes 50 seconds, so his average speed for the race is $\frac{100 \text{ m}}{50 \text{ s}} = 2$ m/s.

The swimmer's change in position for the race is 0 because he ends in the same place he started, so his average velocity for the race is $\frac{0 \text{ m}}{50 \text{ s}} = 0$ m/s.

(c) *What was the swimmer's average speed over the first 20 seconds of the race? Average velocity?*

Solution. In the first 20 seconds, the swimmer swam a distance of 50 m. So, his average speed on this interval was $\frac{50 \text{ m}}{20 \text{ s}} = 2.5$ m/s.

The swimmer's average velocity over the first 20 seconds is $\frac{\text{change in position}}{\text{change in time}} = \frac{s(20)-s(0)}{20-0} = \frac{50 \text{ m}}{20 \text{ s}} =$ 2.5 m/s.

(d) *What was the swimmer's average speed over the last 50 m of the race? Average velocity?*

Solution. The swimmer swam the last 50 m in 30 seconds, so his average speed was $\frac{50 \text{ m}}{30 \text{ s}} = \frac{5}{3}$ m/s.

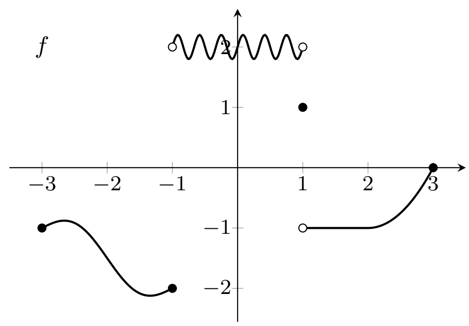
His average velocity on this interval was $\frac{s(50)-s(20)}{50-20} = \frac{-50 \text{ m}}{30 \text{ s}} = -\frac{5}{3}$ m/s.

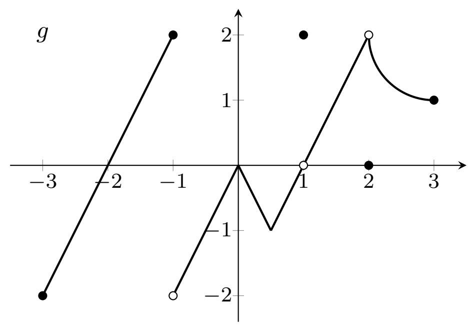
(e) *What is the difference between velocity and speed?*

Solution. As we said already, average velocity over an interval is $\frac{\text{change in position}}{\text{change in time}}$, while average speed is $\frac{\text{change in distance traveled}}{\text{change in time}}$. Just knowing the value of one of these quantities does not tell us the other; for example, in (b), knowing that the average speed for the race is 2 m/s doesn't tell us what the average velocity is, and knowing that the average velocity is 0 m/s doesn't tell us the swimmer's average speed.

On the other hand, instantaneous speed is simply the absolute value of instantaneous velocity.

44. The graphs of f and g are given below.





Evaluate each of the following limits.

(a) $\lim_{x \to 2} (g(x) - f(x))$

Solution. As $x \to 2$, $f(x) \to -1$ and $g(x) \to 2$, so $\lim_{x \to 2}(g(x) - f(x)) = \lim_{x \to 2}(2+1) = \boxed{3}$.

(b) $\lim_{x \to 0} \dfrac{f(x)}{g(x)}$

Solution. As $x \to 0$, $f(x) \to 2$ and $g(x) \to 0$. In this situation (where the denominator tends to 0 but the numerator doesn't), we need to think about the sign of the denominator. Here, $g(x)$ is negative for x near 0, so $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \boxed{-\infty}$. (After all, for x very close to 0, $f(x)$ is very close to 2, and $g(x)$ is a very slightly negative number. When you divide a number close to 2 by a slightly negative number, you get a very negative result.)

(c) $\lim_{x \to -2} \dfrac{f(x)}{g(x)}$

Solution. As $x \to -2$, $f(x)$ approaches $f(-2)$, which is a negative number (although we can't tell its exact value easily from the graph), while $g(x)$ approaches 0. So, we should again think about the sign of the denominator. Here, the sign changes at $x = -2$. For x just a bit less than -2, $g(x)$ is negative, so $\lim_{x \to -2^-} \dfrac{f(x)}{g(x)} = \infty$. For x just a bit bigger than -2, $g(x)$ is positive, so $\lim_{x \to 2^+} \dfrac{f(x)}{g(x)} = -\infty$. Therefore, $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$ $\boxed{\text{does not exist}}$.

(d) $\lim_{x \to 1} \dfrac{f(x)}{g(x)}$

Solution. As $x \to 1$, $f(x)$ does not approach a limit because the one-sided limits don't agree. Therefore, we should look at the given limit from both sides. As $x \to 1^-$, $f(x) \to 2$ and

$g(x) \to 0^-$,[(2)] so $\frac{f(x)}{g(x)} \to -\infty$. As $x \to 1^+$, $f(x) \to -1$ and $g(x) \to 0^+$, so $\frac{f(x)}{g(x)} \to -\infty$. Since $\lim_{x\to 1^-} \dfrac{f(x)}{g(x)}$ and $\lim_{x\to 1^+} \dfrac{f(x)}{g(x)}$ are both $-\infty$, $\lim_{x\to 1} \dfrac{f(x)}{g(x)} = \boxed{-\infty}$.

(e) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

Solution. As $x \to 2$, $f(x) \to -1$ and $g(x) \to 2$, so $\dfrac{f(x)}{g(x)} \to \boxed{-\dfrac{1}{2}}$.

45. *Working algebraically with limits.* Can you evaluate the following limits?

(a) $\lim_{x\to \frac{\pi}{3}} \dfrac{\sin x}{\cos x}$

Solution.

$\lim_{x\to \frac{\pi}{3}} \frac{\sin x}{\cos x} = \boxed{\sqrt{3}}$

(b) $\lim_{x\to 3\pi^+} \dfrac{\sin x}{\cos x}$

Solution.

$\lim_{x\to 3\pi^+} \frac{\sin x}{\cos x} = \boxed{0}$

(c) $\lim_{x\to 3\pi^+} \dfrac{\cos x}{\sin x}$

Solution. $\lim_{x\to 3\pi^+} \dfrac{\cos x}{\sin x} = \boxed{\infty}$.

(d) $\lim_{x\to 3\pi} \dfrac{\cos x}{\sin x}$

Solution. $\lim_{x\to 3\pi} \dfrac{\cos x}{\sin x}$ $\boxed{\text{does not exist}}$.

(e) $\lim_{x\to 2} \dfrac{(\sin x) - 2}{(x-2)^2}$

Solution. $\lim_{x\to 2} \dfrac{(\sin x) - 2}{(x-2)^2} = \boxed{-\infty}$.

(f) $\lim_{x\to 3^+} \ln(x^2 - 9)$

Solution. $\lim_{x\to 3^+} \ln(x^2 - 9) = \boxed{-\infty}$.

In this problem, we'll look at $\lim_{x\to 0} x^2 \sin\left(\dfrac{\pi}{x}\right)$.

46. (a) *Aberforth is thinking about this limit and says, "As $x \to 0$, x^2 approaches 0. 0 times anything is 0, so $\lim_{x\to 0} x^2 \sin\left(\dfrac{\pi}{x}\right)$ must be 0." What do you think of Aberforth's reasoning?*

(2) Remember that writing, "As $x \to 1^-$, $g(x) \to 0^-$" is shorthand for "As $x \to 1^-$, $g(x)$ approaches 0 from below." That is, when x is a bit less than 1, $g(x)$ is a bit less than 0.

Solution. Aberforth's reasoning is incorrect, which we can see from an example like $\lim_{x\to 0} x^2 \cdot \frac{1}{x^4}$. Aberforth would think that this limit is 0, but we can use algebra to rewrite it as $\lim_{x\to 0} \frac{1}{x^2}$, which is ∞.

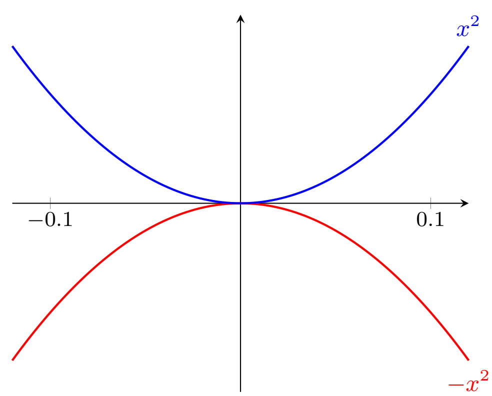
In general, it's never safe to look at just one piece of a limit; you always need to look at all pieces of a limit to understand it!

(b) *Find* $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$. *Explain your reasoning carefully.*

Solution. Aberforth's mistake was ignoring the $\sin\left(\frac{\pi}{x}\right)$ piece of this limit, so let's try to understand that piece. We've seen in **??** that it doesn't approach a limit as $x \to 0$ and that it actually behaves very wildly near $x = 0$. However, one simple thing we can say about it is that $\sin\left(\frac{\pi}{x}\right)$ is always between -1 and 1, just because sine of *anything* is between -1 and 1. Therefore, $x^2 \sin\left(\frac{\pi}{x}\right)$ is always between $x^2(-1) = -x^2$ and $x^2(1) = x^2$. That is

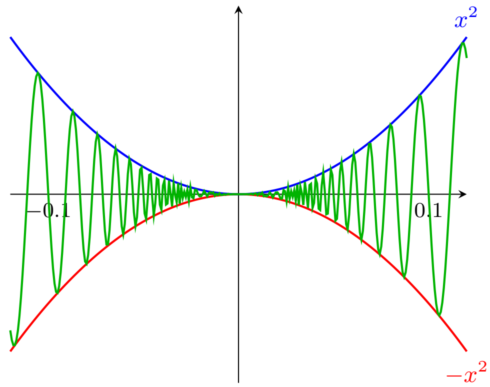
$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2$$

But if we graph $-x^2$ and x^2, we see that any function that is stuck between them must tend to 0 as $x \to 0$, because both x^2 and $-x^2$ tend to 0 as $x \to 0$:



(c) *Sketch a rough graph of* $f(x) = x^2 \sin\left(\frac{\pi}{x}\right)$ *for* x *near 0.*

Solution. As we saw in **??**, around $x = 0$, $\sin\left(\frac{\pi}{x}\right)$ oscillates (wildly!) between -1 and 1. Therefore, $x^2 \sin\left(\frac{\pi}{x}\right)$ oscillates between $x^2(-1) = -x^2$ and $x^2(1) = x^2$. Its graph is the green one below:



47. Evaluate the limits below. If the limit does not exists, write whether it is ∞ or $-\infty$. If it does not exists and also not $\pm\infty$, explain why the limit does not exist

(a) $\lim_{x\to 0} \dfrac{3x - x\cos x}{\sin 2x}$.

Solution. $\lim_{x\to 0} \frac{3x - x\cos x}{\sin 2x} = 1$

(b) $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$.

Solution. The limit does not exist, and there is nothing more we can say.

(c) $\lim_{x\to\infty} 2x - \sqrt{4x^2+1}$.

Solution.

$\lim_{x\to\infty} 2x - \sqrt{4x^2+1} = 0$

(d) $\lim_{x\to 0} \csc(2x)\tan(3x)$

Solution. $\lim_{x\to 0} \csc(2x)\tan(3x) = \frac{3}{2}$

(e) $\lim_{x\to 1} \dfrac{x^2+x-2}{x^2-3x+2}$

Solution.

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-3x+2} = -3$$

(f) $\lim_{x\to\infty} \dfrac{\sqrt{2x^2+2x+3}}{3x-1}$

Solution.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+2x+3}}{3x-1} = \frac{\sqrt{2}}{3}$$

(g) $\lim_{x\to\infty} \dfrac{e^{2x}+e^x+4}{7e^{2x}+e^x-4}$

Solution. $\lim_{x\to\infty} \frac{e^{2x}+e^x+4}{7e^{2x}+e^x-4} = \boxed{\frac{1}{7}}$

(h) $\lim_{x\to-\infty} \dfrac{e^{2x}+e^x+4}{7e^{2x}+e^x-4}$

Solution. You might see this, and immediately jump to our strategy of dividing the numerator and denominator by the largest term in the denominator. However, that wouldn't work here! That's a strategy that only works with limits of the form $\left(\frac{\infty}{\infty}\right)$. Here, $x \to -\infty$, $e^x \to 0$, and $e^{2x} \to 0$. So the numerator $\to 4$, and the denominator $\to -4$. That is

$$\lim_{x\to-\infty} \frac{e^{2x}+e^x+4}{7e^{2x}+e^x-4} = \frac{4}{-4} = \boxed{-1}$$

.

(i) $\lim_{x\to\infty} \dfrac{70^{x/2}-1{,}000{,}000}{2^{3x}+3^{2x}+10^{100}}$

Solution. $\lim_{x\to\infty} \frac{70^{x/2}-1{,}000{,}000}{2^{3x}+3^{2x}+10^{100}} = \boxed{0}$

(j) $\lim_{x\to-\infty} \sin\left(\dfrac{1}{x}\right)$

Solution. Note that as $x \to -\infty$, we have that $\frac{1}{x} \to 0$. Since sine of a number really close to 0, is a number really close to 0, we have that $\lim_{x\to-\infty} \sin\left(\dfrac{1}{x}\right) = \boxed{0}$.

48. *Let a be a constant and f be the function defined by*

$$f(x) = \begin{cases} \frac{ax}{6} & \text{for } x < 1 \\ \frac{1}{x+a} & \text{for } x \geq 1 \end{cases}.$$

Find all values of a for which f is continuous on $(-\infty, \infty)$. (Do this without using a calculator!)

Solution. The only value of a that works is $\boxed{a = 2}$.

49. Let f be the function defined by

$$f(x) = \begin{cases} -x^2+1 & \text{for } x \leq 1 \\ ax+b & \text{for } x > 1 \end{cases}$$

(a) *For what values of a and b will f be continuous at $x = 1$?*

Solution. If we want the function to be continuous at $x = 1$, we need to make sure that

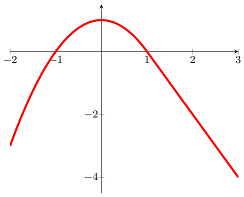
$$\lim_{x\to 1} f(x) = f(1)$$

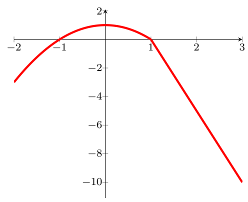
For the limit to exist, the left hand limit needs to equal the right hand limit.

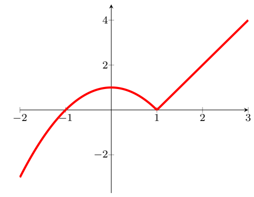
$$\lim_{x\to 1^+} f(x) = a + b$$

$$\lim_{x\to 1^-} f(x) = 0$$

Therefore, as long as $a = -b$ the function is continuous. Lets draw a few examples:







Note that based off the pictures, it makes sense that there are infinitely many solutions to consider here!

(b) *Are there any values of a and b for which f will be differentiable at $x = 1$?*

Solution. We want to avoid having a sharp corner at $x = 1$. To accomplish this we want the derivatives of the two pieces to match up.

$$\frac{d}{dx}(-x^2 + 1) = -2x$$

$$\frac{d}{dx}(ax + b) = a$$

At $x = 1$ we need $-2x = a$, so $a = -2$. This graph is pictured on the left above. Again, thinking graphically, it makes sense why now we should expect a unique solution.

50. For each of the following functions, do the following:

- Use the **limit definition of the derivative** to compute the slope at the given point. No credit will be given for using any other method.
- Use your answer for the slope to find the equation of the tangent line at the given point.

(a) $f(x) = \sin x$ *at* $x = 0$.

Solution. By the definition of the derivative,

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{\sin h}{h} \\ &= \boxed{1} \end{aligned}$$

Slope of the tangent line at $x = 0$ is $f'(0) = 1$. Since, the tangent line is passing through the point $(0,f(0)) = (0, 0)$, the equation of the tangent line at $x = 0$ is $\boxed{y = x}$.

(b) $f(x) = \dfrac{1 - x}{x}$ *at* $x = 1$.

Solution. By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1-x-h}{x+h} - \frac{1-x}{x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h} - 1 - \frac{1}{x} + 1}{h} \\
&= \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)x}}{h} \\
&= \lim_{h\to 0} \frac{-h}{hx(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= -\frac{1}{x^2}
\end{aligned}$$

Slope of the tangent line is $f'(1) = \boxed{-1}$. The tangent line passing through the point $(1,f(1)) = (1,0)$. So the equation of the tangent line at $x = 1$ is $\boxed{y = -(x-1)}$.

51. Consider the function $f(x) = \dfrac{1}{\sqrt{x+1}}$

(a) *Find $f'(x)$ using the formal definition of the derivative.*

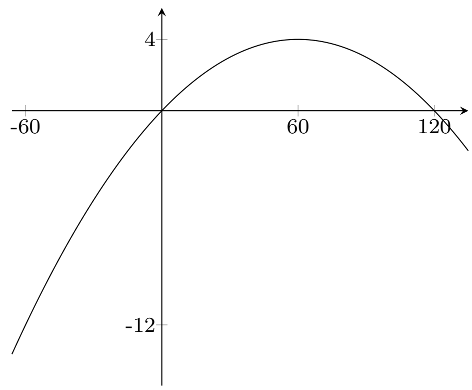
Solution. By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{(x+h)+1}} - \frac{1}{\sqrt{x+1}}}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{\sqrt{(x+h)+1}} - \frac{1}{\sqrt{x+1}} \right) \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{h\sqrt{(x+h)+1}} - \frac{1}{h\sqrt{x+1}} \right) \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x+1}}{h\sqrt{x+h+1}\sqrt{x+1}} - \frac{\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \right) \\
&= \lim_{h\to 0} \frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \cdot \frac{\sqrt{x+1}+\sqrt{x+h+1}}{\sqrt{x+1}+\sqrt{x+h+1}} \\
&= \lim_{h\to 0} \frac{(x+1)-(x+h+1)}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)} \\
&= \frac{-1}{\sqrt{x+0+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+0+1}\right)} \\
&= \boxed{-\frac{1}{2(x+1)^{3/2}}}
\end{aligned}$$

(b) *Find x so that the tangent line to f at x has slope $-\frac{1}{16}$.*

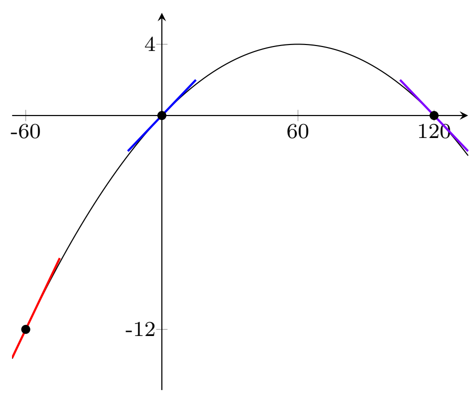
Solution. We just need to know when $f'(x) = -\frac{1}{16}$. This happens at $x = \boxed{3}$

52. One day, a hitchhiker is wandering along a highway near a gas station. Suppose his position at time t is $f(t) = 360t - 3t^2$, where t is measured in minutes after noon and the position is given in feet east of the gas station. Here is the graph of $f(t)$:



(a) *On the graph above, sketch slopes that represent the hitchhiker's instantaneous velocity at 11 am, 12:30 pm, and 2 pm. Which of these velocities is the biggest?*

Solution. The instantaneous velocity is given by the slope of the tangent line. The instantaneous velocity at 11 am ($t = -60$) is the slope of the red line. The instantaneous velocity at noon ($t = 0$) is the slope of the blue line. The instantaneous velocity at 2 pm ($t = 120$) is the slope of the purple line. The instantaneous velocity is biggest when the absolute value of slope is steepest. This happens at $t = -60$ or at 11 am. Note that as you go along a concave down graph, the slope decreases from left to right.



(b) *Were there any times when the hitchhiker's instantaneous velocity was 0? If so, when?*

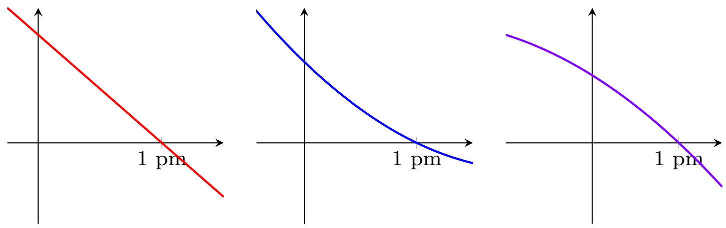
Solution. Yes! The slope is zero at $t = 60$ (or 1 pm) as the tangent line is horizontal at that point. So the instantaneous velocity is 0 at 1 pm.

(c) *At what times was the hitchhiker's instantaneous velocity positive?*

Solution. The instantaneous velocity is positive when the slope is positive. This happens at all times before 1 pm.

(d) *Sketch a rough graph of the hitchhiker's velocity as a function of time.*

Solution. Note that from our work above we see that before 1 pm, the velocity is positive (because $f(t)$ is increasing, so the slope is positive) and decreasing (because $f(t)$ is concave down so the slope is decreasing). After 1 pm the velocity is negative (because $f(t)$ is decreasing, so the slope is negative) and decreasing (because $f(t)$ is concave down so the slope is decreasing). At 1 pm, it is 0. Here are three possible sketches of the velocity:



(A) (B) (C)

So we see that the velocity, which was the derivative of the position function, is a function in its own right!

(e) *Using the definition of the derivative, calculate the hitchhiker's velocity at time t. This is denoted by $f'(t)$ or $\frac{df}{dt}$. Does this agree with your sketch?*

Solution. The instantaneous velocity is the derivative of the position function. This is denoted by $f'(t)$ or $\frac{df}{dt}$. We have that

$$\frac{df}{dt} = f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

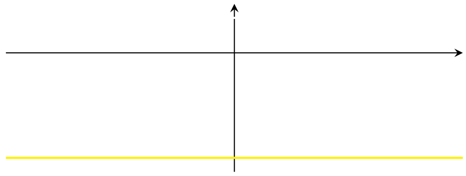
Let's compute this:

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \frac{360(t+h) - 3(t+h)^2 - (360t - 3t^2)}{h} \\ &= \lim_{h\to 0} \frac{360(t+h) - 3(t+h)^2 - 360t + 3t^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{360t} + 360h - \cancel{360t} + 3t^2 - 3(t+h)^2}{h} \\ &= \lim_{h\to 0} \frac{360h + 3(t^2 - (t+h)^2)}{h} \\ &= \lim_{h\to 0} \frac{360h + 3((-h)(2t+h))}{h} \\ &= \lim_{h\to 0} \frac{360h - 3h(2t+h)}{h} \\ &= \lim_{h\to 0} (360 - 3(2t+h)) \\ &= 360 - 6t \end{aligned}$$

So $v(t) = \frac{df}{dt} = f'(t) = 360 - 6t$. Ah-ha! Now we see that indeed this is a line with negative slope (hence decreasing), and so graph (A) in red is the correct velocity graph!

(f) *What is the hitchhiker's acceleration at time t? Sketch a graph of his acceleration as a function of time.*

Solution. Acceleration is the derivative of velocity. That is $a(t) = v'(t)$. Since $v(t)$ is a straight line, we know exactly what its slope is! The slope of the line $v(t) = 360 - 6t$ is -6. And so $a(t) = -6$. Here is the graph:



53. Sara, the owner of a cupcake truck, has been experimenting with the price of the cupcakes she sells. Unsurprisingly, she has found that the number of cupcakes she sells per day depends on the price. Let $C(p)$ be the number of cupcakes she sells when the price of a cupcake is p cents.

    So far, Sara has found that, when the price of a cupcake is 300 cents, she sells 600 cupcakes a day.

    (a) *What are the units of $C'(300)$?*

    Solution. Since C is in cupcakes and p is in cents, $C'(300)$ is in $\boxed{\text{cupcakes per cent}}$. One way to understand this is by going back to the definition of the derivative: $C'(300) = \lim_{h \to 0} \frac{C(300+h) - C(300)}{h}$; the numerator $C(300+h) - C(300)$ is in cupcakes, and the denominator h is in cents. Alternatively, you could think about how we represent $C'(300)$ graphically: when we graph $C(p)$ (with units of cents on the horizontal axis and cupcakes on the vertical axis), it's a slope, so it must have units of cupcakes per cent.

    (b) *Do you expect $C'(300)$ to be positive or negative? Why?*

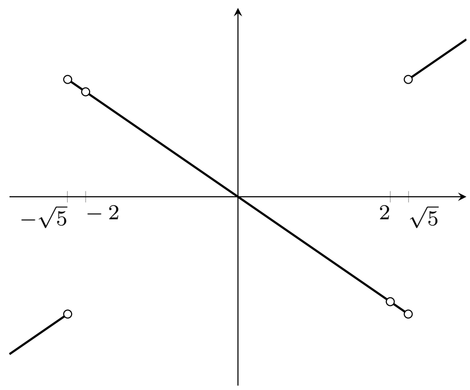
    Solution. We expect Sara to sell fewer cupcakes when the price is higher, so we expect $C(p)$ to be a decreasing function. Therefore, $C'(300)$ should be $\boxed{\text{negative}}$. (Tangent lines for a decreasing function will have negative slope.)

    (c) *Suppose $C'(300) = -5$. If Sara raises the price of cupcakes to 310 cents, how many cupcakes do you expect her to sell per day?*

    Solution. Saying $C'(300) = -5$ means that, when the price of cupcakes is 300 cents, the instantaneous rate of change in the number of cupcakes Sara sells with respect to price is -5 cupcakes per cent. In other words, for each cent Sara raises the price above 300, we expect her to sell 5 fewer cupcakes. Therefore, when she raises the price from 300 to 310 (an increase of 10 cents), we expect her to sell 50 fewer cupcakes, so we expect her to sell about $600 - 50 = \boxed{550}$ cupcakes.

54. *Let $f(x) = \left| \frac{x^4 - 9x^2 + 20}{x^2 - 4} \right|$. Sketch the graph of f, and then sketch the graph of f'.*

    Solution. The derivative of f looks like this:



55. Gromit has been growing a giant squash for Tottington Hall's annual Giant Vegetable Competition. He has carefully tracked his squash's length and weight. Let $w(\ell)$ be the squash's weight in kg when its length is ℓ cm.

(a) *What are the units of $w'(\ell)$?*

Solution. $\boxed{\frac{\text{kg}}{\text{cm}}}$. The derivative is the rate of change of $w(l)$ with respect to l and we know the units of $w(l)$ are in kg and the units of l are cm.

(b) *Another notation for $w'(\ell)$ is $\frac{dw}{d\ell}$. Do you expect $\frac{dw}{d\ell}$ to be positive or negative? Why?*

Solution. $\frac{dw}{d\ell}$ should be positive. As the length of the squash increases, the mass should increase.

(c) Suppose $w'(70) = 8$. (This statement can also be written as $\left.\frac{dw}{d\ell}\right|_{\ell=70} = 8$.) Which is the following is the most reasonable conclusion?

A. It takes 70 days for the squash to grow to be 8 kg.

B. When the squash is 70 cm long, it weighs 8 kg.

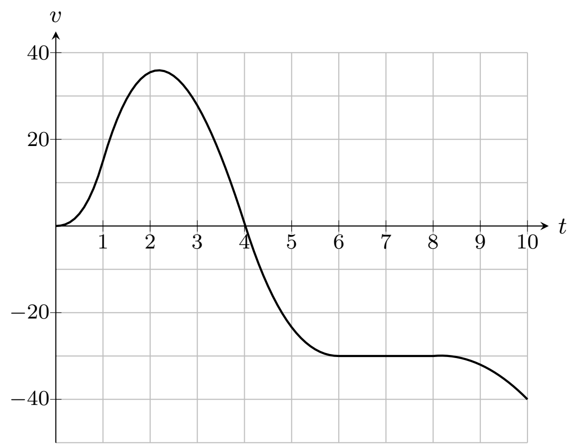
C. When the squash is 70 cm longer than its current length, it will weigh 8 kg more than it currently does.

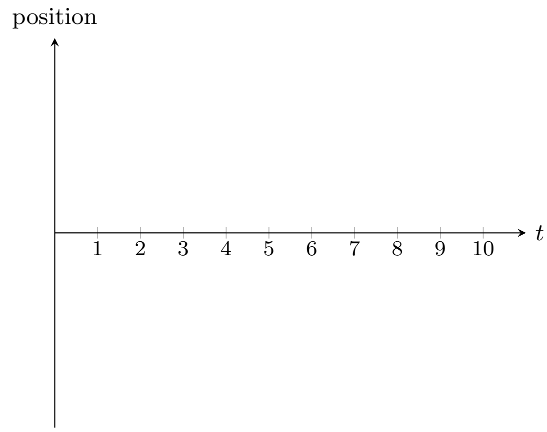
(D.) When the squash grows from 70 cm to 71 cm, it will gain about 8 kg in weight. This one.

(d) *Interpret the statement $w'(105) = 10$ in words.*

Solution. When the squash grows from 105 cm to 106 cm, it will gain *about* 10 kg in mass.

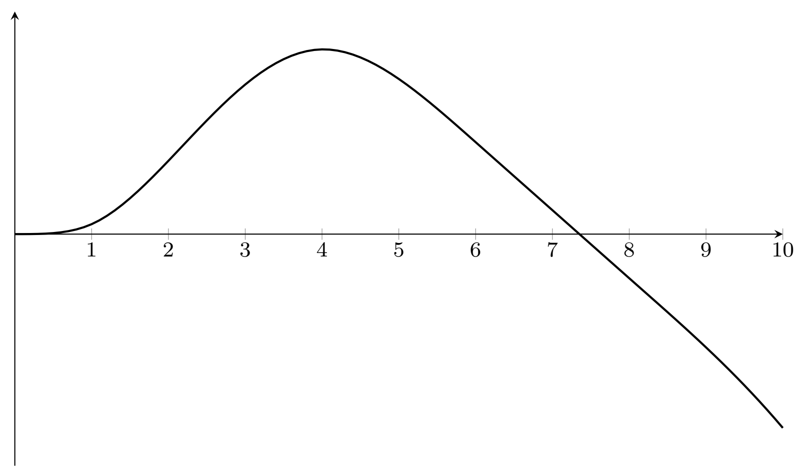
56. Albert is taking a very long car trip. Suppose that the following graph gives his velocity v (measured in mph) as a function of time (measured in hours).





(a) *At time 0, Albert is at position 0. On the axes above, sketch a graph of his position $A(t)$ as a function of time.*

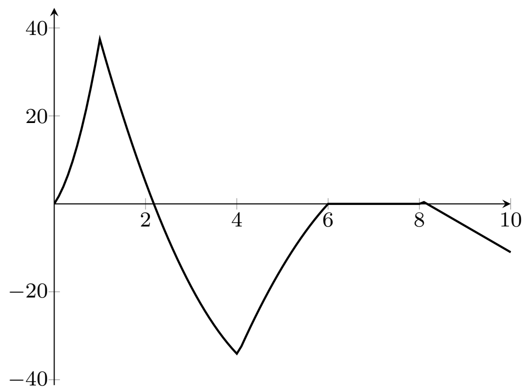
Solution. Here is a reasonable graph of Albert's position:



Note that the position graph we've drawn crosses the horizontal axis, but we don't know how to tell exactly where this happens. (This is something that will get resolved when you take 1552!) However, it seems reasonable that Albert would go past his original starting position; after all, his total travel in the positive direction lasts 4 hours, with most of it happening under 30 mph. His travel in the negative direction includes 4 hours where he's traveling with a speed of at least 30 mph, so he travels further in the negative direction than in the positive direction.

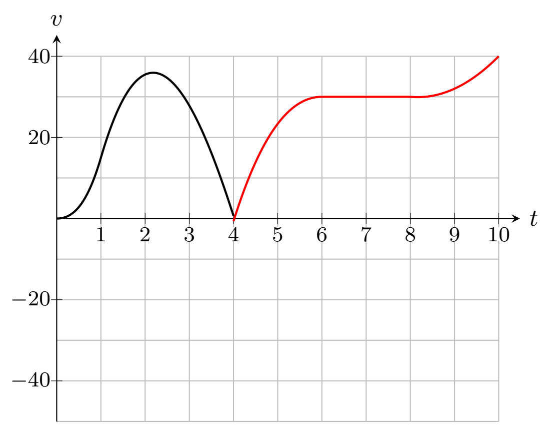
(b) *Sketch the graph of $A''(t)$; what does it represent?*

Solution. $A''(t)$ represents Albert's acceleration at time t. It is the derivative of Albert's velocity, the graph we were given originally. Here's a sketch of its graph:



(c) *Let $S(t)$ denote Albert's speed at time t. Sketch a graph of $S(t)$.*

Solution. Speed can never be negative. Can you imagine your speedometer giving you negative readings?! Instantaneous speed is just the absolute value of instantaneous velocity. So to get the graph of $S(t)$, we can simply flip over the x-axis, any pieces where the velocity is negative. Here is the graph we'd get. The red piece is the piece flipped over:



57. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \left(3e^u - u^2 + 5\right) du$

Solution. $\displaystyle\int \left(3e^u - u^2 + 5\right) du = \boxed{3e^u - \frac{u^3}{3} + 5u + C}.$

(b) $\displaystyle\int \frac{5}{7x}\, dx$

Solution. First, this is an indefinite integral, so it's really asking for all antiderivatives of $\dfrac{5}{7x}$. Let's rewrite this: $\displaystyle\int \frac{5}{7x}\, dx = \int \frac{5}{7} \cdot \frac{1}{x}\, dx = \boxed{\frac{5}{7}\ln|x| + C}.$

(c) $\displaystyle\int (e^\pi + ey + \pi^y)\, dy$

Solution. $\int (e^\pi + ey + \pi^y)\,dy = \boxed{e^\pi y + \frac{e}{2}y^2 + \frac{\pi^y}{\ln \pi} + C}$. (Keep in mind that e^π is a constant.)

(d) $\int (\pi + x)\sqrt{x}\,dx$

Solution. We don't have a rule about integrating products, so let's first rewrite the integrand so that it's no longer a product:

$$\begin{aligned}\int (\pi + x)\sqrt{x}\,dx &= \int (\pi\sqrt{x} + x\sqrt{x})\,dx \\ &= \int \left(\pi x^{1/2} + x^{3/2}\right) dx \\ &= \boxed{\frac{2}{3}\pi x^{3/2} + \frac{2}{5}x^{5/2} + C}\end{aligned}$$

(e) $\int \left(\frac{e}{x^2} + 2^x \cdot 3^x\right) dx$

Solution. We don't have a rule about integrating products, so we must first rewrite $2^x \cdot 3^x$:

$$\begin{aligned}\int \left(\frac{e}{x^2} + 2^x \cdot 3^x\right) dx &= \int \left(ex^{-2} + 6^x\right) dx \\ &= -ex^{-1} + \frac{1}{\ln 6}6^x + C \\ &= \boxed{-\frac{e}{x} + \frac{1}{\ln 6}6^x + C}\end{aligned}$$

58. True/False

(a) Let $f(x)$ be continuous on the interval $[-1,3]$. If $f(-1) = 2$ and $f(3) = 8$, then by the Intermediate Value Theorem, $f(x)$ cannot have a zero on $[-1,3]$.

Solution. False.

(b) $\arccos(\cos(-\frac{\pi}{8})) = \frac{\pi}{8}$.

Solution. True. Since $\cos(x)$ is an even function, $\cos(-\frac{\pi}{8}) = \cos(\frac{\pi}{8})$. Since the range of $\arccos(x)$ is $[0,\pi]$, we have $\arccos(\cos(-\frac{\pi}{8})) = \frac{\pi}{8}$.

(c) Let f and g be two functions defined on $(-\infty, \infty)$, $f(0) \neq g(0)$, then we must have $\lim_{x\to 0} f(x) \neq \lim_{x\to 0} g(x)$.

Solution. False.

(d) If f is even and $\lim_{x\to 0^+} f(x)$ exists, then $\lim x \to 0 f(x)$ exists.

Solution. True.

(e) If $g(x)$ is an odd function and has a local maximum at $x = c$, then it must have a local minimum at $x = -c$.

Solution. True. Since $g(-x) = -g(x)$, g is symmetric about the origin. If $x = c$ is a local maximum, then for x close to c, $g(x) \leq g(c)$. So for x close to $-c$, $g(x) \geq g(-c)$. So $x = -c$ is a local minimum

(f) If $f(x)$ is concave down and $f'(a) = 0$, then f has a maximum $x = a$.

Solution. True, because of the second derivative test.

(g) If $f'(a) = 0$, then $f(x)$ has either a local maximum or a local minimum at $x = a$.

Solution. False, consider $f(x) = x^3$ and $a = 0$.

(h) If c is a local minimum of f, then $f''(c) > 0$.

Solution. False. f'' could be 0 or undefined at $x = 0$, making the second derivative test inconclusive.

(i) A function may have three different horizontal asymptotes.

Solution. False. We find horizontal asymptotes of a function $f(x)$ by computing $\lim_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Thus, a function can have at most two horizontal asymptotes.

(j) The function $|x - 2|$ is differentiable at $x = 2$.

Solution. False, the graph has a corner (is not locally linear) at $x = 2$.

(k) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$.

Solution. False. Consider $f(x) = x$ and $g(x) = x + 1$.

(l) If f is differentiable at $x = a$, then f is continuous at $x = a$.

Solution. True. Differentiability requires continuity in the definition.

(m) If f is continuous at $x = a$, then f is differentiable at $x = a$.

Solution. False. Corners (points where a function is not locally linear) are points where a function is continuous but not differentiable. For example $|x|$ is continuous but not differentiable at $x = 0$.

(n) There is a function f so that $f(x) > 0$ for all x but $f'(x) < 0$ for all x.

Solution. True. For example $f(x) = e^{-x}$. Or sketch any curve that decreases to a nonnegative horizontal asymptote.