

MATH 1554 READING DAY STUDY SESSION - FALL 2017

1. A  $5 \times 4$  matrix  $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \vec{a}_4]$  has all non-zero columns, and  $\vec{a}_4 = 2\vec{a}_1 + 3\vec{a}_2 + 5\vec{a}_3$ . Find a non-trivial solution to  $A\vec{x} = \vec{0}$ .
2. For what values of  $h$ , if any, are the columns of  $A$  linearly dependent?

$$A = \begin{bmatrix} 1 & 0 & h \\ 0 & 1 & 1 \\ h & 1 & 0 \end{bmatrix}$$

3. For what values of  $h$  is  $\vec{b}$  in the plane spanned by  $\vec{a}_1$  and  $\vec{a}_2$ ?

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

4. Express the solution to  $A\vec{x} = \vec{0}$  in parametric vector form, where

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

5. Write down the standard matrix  $A$  of  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with  $T(\vec{x}) = -\vec{x}$ .

6. Find the domain and codomain of the linear transformation  $T$  given by the standard matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 5 & 7 & 3 \\ 2 & 5 & -1 \end{bmatrix}$$

Is this linear transformation one-to-one? Is it onto?

7. Let  $A = \begin{bmatrix} -5 & 2 \\ -1 & -3 \end{bmatrix}$ . Find its eigenvalue(s) and find an invertible matrix  $P$  and a (rotation-scaling) matrix  $C$  such that  $A = PCP^{-1}$ .

8.  $W$  is the set of all vectors of the form  $\begin{bmatrix} x \\ x+y \\ y \end{bmatrix}$ . Which of the following vectors are in  $W^\perp$ ?

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

9. Identify all values of  $a, b$ , and  $c$ , if any, so that the columns of  $U$  are mutually orthogonal.

$$U = \begin{bmatrix} 3 & 2 & 2 \\ -4 & 1 & b \\ 2 & a & c \end{bmatrix}$$

10. Use the Gram-Schmidt process to construct an orthonormal basis of the column space of  $A$ .

11. Let  $A = QR$ , where

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ -2 & -2 \end{bmatrix}, \quad Q = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}$$

Compute the upper triangular matrix  $R$ .

12. Give an example of a  $2 \times 2$  matrix that is in echelon form, is orthogonally diagonalizable, but is not invertible.

13. True or False?

- (i) If the set of vectors  $\{\vec{u}, \vec{v}, \vec{w}\}$  is linearly independent, so is every pair of vectors  $\{\vec{u}, \vec{v}\}$ ,  $\{\vec{u}, \vec{w}\}$ , and  $\{\vec{v}, \vec{w}\}$ .
- (ii) If every pair of vectors  $\{\vec{u}, \vec{v}\}$ ,  $\{\vec{u}, \vec{w}\}$ , and  $\{\vec{v}, \vec{w}\}$  is linearly independent, so is the set of vectors  $\{\vec{u}, \vec{v}, \vec{w}\}$ .
- (iii) For any two vectors  $\vec{u}$  and  $\vec{v}$ , we have  $Span\{\vec{u}, \vec{v}\} = Span\{\vec{u}, 2\vec{u} + 3\vec{v}, 4\vec{v}\}$ .
- (iv) If  $\vec{u}$  and  $\vec{v}$  are two distinct nonzero vectors, then there are exactly two vectors in  $Span\{\vec{u}, \vec{v}\}$ .
- (v) The transformation given by  $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_2 \end{bmatrix}$  is linear.
- (vi) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a projection onto the  $x_1$ -axis. The range of  $T$  is  $\mathbb{R}^2$ .
- (vii) The transformation given by  $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 1 \\ x_2 \end{bmatrix}$  is linear.
- (viii) A linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  can be onto.
- (ix) The composition  $S \circ T$  of two one-to-one linear maps is one-to-one.
- (x) The range of a one-to-one linear map  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  may be a line.
- (xi) The eigenvalues of a square matrix  $A$  are the same as the eigenvalues of its reduced row echelon form.
- (xii) If  $\vec{u}$  and  $\vec{v}$  are eigenvectors corresponding to the same eigenvalue  $\lambda$ , then every linear combination of  $a\vec{u} + b\vec{v}$  with  $a, b \in \mathbb{R}$  (except the zero vector) is an eigenvector.
- (xiii) The geometric multiplicity of an eigenvalue is less than or equal to the algebraic multiplicity.
- (xiv) All upper triangular  $3 \times 3$  stochastic matrices are not regular.
- (xv) If  $A$  is a diagonalizable matrix, then  $\lambda = 0$  is not an eigenvalue of  $A$ .
- (xvi) An  $n \times n$  matrix with  $n$  distinct eigenvalues is diagonalizable.
- (xvii) If complex  $\lambda$  is an eigenvalue, then so is  $-\lambda$ .
- (xviii) Every real  $3 \times 3$  matrix must have a real eigenvalue.
- (xix) For any three vectors  $\vec{x}$ ,  $\vec{y}$ , and  $\vec{z}$  we have  $(\vec{x} \cdot \vec{y})\vec{z} = (\vec{y} \cdot \vec{z})\vec{x}$ .
- (xx) Let  $\vec{x} \cdot \vec{y} > 0$ . Then the angle between  $\vec{x}$  and  $\vec{y}$  is less than  $90^\circ$ .
- (xxi) Every orthogonal set of nonzero vectors  $\{\vec{x}, \vec{y}, \vec{z}\}$  is linearly independent.
- (xxii) Let  $\hat{y}$  be the orthogonal projection of a vector  $\vec{y}$  onto the subspace  $W \subset \mathbb{R}^n$ . Then the transformation  $T(\vec{y}) = \hat{y}$  is linear.
- (xxiii) The inverse of an orthogonal matrix  $Q$  equals  $Q^T$ .
- (xxiv) A least square solution  $\hat{x}$  to  $A\vec{x} = \vec{b}$  always satisfies  $A\vec{x} = \vec{b}$ .
- (xxv) A least square solution  $\hat{x}$  to  $A\vec{x} = \vec{b}$  minimizes the distance  $\|A\vec{x} - \vec{b}\|$ . That is, the distance is the shortest for  $\vec{x} = \hat{x}$ .
- (xxvi) A  $n \times n$  symmetric matrix  $A$  will always have  $n$  real and distinct eigenvalues.
- (xxvii) A  $n \times n$  symmetric matrix  $A$  will have algebraic multiplicity = geometric multiplicity for each of its eigenvalues.
- (xxviii) If a matrix  $A$  is orthogonally diagonalizable, then  $A^k$  is also orthogonally diagonalizable for all  $k \in \mathbb{Z}^+$ .
- (xxix) The eigenvalues of  $A^T A$  are always real for any  $m \times n$  matrix  $A$ .
- (xxx) A negative definite matrix cannot be invertible.
- (xxxii) Matrices  $A$  and  $A^T$  have the same non-zero singular values.
- (xxxii) For any matrix  $A$ ,  $A^t A$  has non-negative, real eigenvalues.

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1.  $\vec{x} = (2, 3, 5, -1)$

2. None. The columns of  $A$  are linearly independent.

3.  $h = 3$

4.  $\vec{x} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.$

5.  $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$

6. Domain:  $\mathbb{R}^3$ , Codomain:  $\mathbb{R}^4$ , Not one-to-one, Not onto.

7.  $\lambda = -4 \pm i, P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} -4 & -1 \\ 1 & -4 \end{bmatrix}$

8.  $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

9.  $a = -1, b = 7, c = 11$

10.  $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \text{col}(A) = \text{Span} \left\{ \frac{\vec{v}_1}{\sqrt{6}}, \frac{\vec{v}_2}{2\sqrt{2}} \right\}$

11.  $R = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$

12.  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

13. .

- |              |             |              |               |
|--------------|-------------|--------------|---------------|
| (i) True     | (ix) True   | (xvii) False | (xxv) True    |
| (ii) False   | (x) False   | (xviii) True | (xxvi) False  |
| (iii) True   | (xi) False  | (xix) False  | (xxvii) True  |
| (iv) False   | (xii) True  | (xx) True    | (xxviii) True |
| (v) False    | (xiii) True | (xxi) True   | (xxix) True   |
| (vi) False   | (xiv) True  | (xxii) True  | (xxx) False   |
| (vii) False  | (xv) False  | (xxiii) True | (xxxi) True   |
| (viii) False | (xvi) True  | (xxiv) False | (xxxii) True  |