Worksheet for Reading Day Study Session Math 1553

1. The following are instructions for a linear algebra game, MatrixToe. It is similar to tic-tac-toe, and is meant for two players.

   - There are two players, the 1-player (1P) and the 0-player (0P).
   - 0P and 1P take turns placing numbers into an empty $N \times N$ matrix
   - the game ends when all the matrix elements have a number.
   - 1P: can only place 1's in the matrix, wins if the matrix is invertible.
   - 0P: can only place 0's in the matrix, wins if the matrix is singular.

   (a) Let $N = 3$. Decide who is the 1P and the 0P, who goes first, play a few games of Matrix Toe, and determine who won for each game.

      a) Game 1: $\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

         Who won? ______ Why?

      b) Game 2: $\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

         Who won? ______ Why?

      c) Game 3: $\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

         Who won? ______ Why?

   (b) Describe at least three strategies that the 0P might use to win.

   (c) If possible, fill in the missing elements of the matrices below with numbers 0 or 1, so that each of the matrices are singular. If it is not possible to do so, state why.

$$A = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

   Hint: you can solve this problem by inspection. You don't need to use row reduction. A few games of MatrixToe may help you see why.

2. If possible, give an example of the following.

   (a) A matrix whose columns form an orthogonal basis for $\mathbb{R}^4$.

   (b) A matrix A that is in echelon form, and

$$\dim\left((\text{Row}(A))^{\perp}\right) = 2$$
$$\dim\left((\text{Col}(A))^{\perp}\right) = 3$$

   (c) A vector $\vec{v} \in \mathbb{R}^3$ and a subspace W such that $\text{proj}_W \vec{v} = \vec{v}$, and $\dim(W) = 2$.

   (d) An orthogonal matrix, in echelon form, whose columns span a 2-dimensional subspace of $\mathbb{R}^3$.

   (e) A matrix C such that the linear system $C\vec{x} = \vec{b}$ is inconsistent but has a unique least-squares solution, where $\vec{x} \in \mathbb{R}^3$ and

$$\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   (f) A subspace S, of $\mathbb{R}^4$, that satisfies $\dim(S) = \dim(S^{\perp}) = 2$.

   (g) Two linearly independent vectors that are orthogonal to $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

   (h) A subspace, S, of $\mathbb{R}^3$ such that $\dim(S^{\perp}) = 2$.

   (i) A 2×3 matrix whose columns are linearly independent.

   (j) A 2×2 matrix that is invertible and does not have an LU decomposition.

   (k) A 2×2 matrix whose eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$, and whose corresponding eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

   (l) An invertible 2×2 matrix whose determinant is zero.

   (m) A 2×2 matrix that is diagonalizable but not invertible.

   (n) A 4×3 matrix in reduced echelon form, whose columns span $\mathbb{R}^4$.

   (o) A 3×3 matrix C, that is in reduced echelon form, has exactly two pivots, and satisfies

$$C \begin{pmatrix} 2 \\ -8 \\ 1 \end{pmatrix} = \vec{0}$$

.

3. Match the items in the column on the left with the items in the column on the right. Some items match to multiple items.

(a) $\text{Proj}_{\vec{x}}\vec{y}$

(b) A set of vectors includes the zero vector.

(c) $\det A \det B$

(d) Every column of A has a pivot

(e) A basis for $\text{Col}(A)$.

(f) U is an orthogonal matrix.

(g) Orthogonal complement $W^{\perp}$

(h) $(\text{Row}A)^{\perp}$

(i) $(\text{Col}A)^{\perp}$

(j) Orthonormal vectors

(k) A is singular

(l) 0 is not an eigenvalue of A

(m) PD^kP^{-1}

(n) A is a 3×4 matrix with linearly independent columns.

(o) Orthogonal projection of $\vec{y}$ onto V

(p) A does not have an LU decomposition

(q) A has the decomposition $A = PDP^{-1}$

(r) T is a linear transformation whose standard matrix, A, is one-to-one.

(I) $\{\vec{x} \,:\, \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$

(II) $\dfrac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}\vec{x}$

(III) Unit length, pairwise orthogonal

(IV) $\det(A) = 0$

(V) Not possible

(VI) $\det(A) \neq 0$

(VII) $\left(PDP^{-1}\right)^k$

(VIII) Row swaps are needed to express A in echelon form.

(IX) $\text{Null}A$

(X) The vector $\widehat{y} \in V$ closest to $\vec{y}$.

(XI) $\text{Null}A^T$

(XII) The eigenvalues of A are distinct.

(XIII) Its columns are orthonormal.

(XIV) The vectors are linearly dependent.

(XV) The system $A\vec{x} = \vec{0}$ has only the trivial solution.

(XVI) The columns of A are linearly independent.

(XVII) The pivot columns of A.

(XVIII) $\det(AB)$