Math 1552: Integral Calculus

Final Exam Study Guide, Fall 2017

1. Evaluate the integrals:

(a) $\int_1^2 \frac{3x-5}{x^3} dx$.

(b) $\int_1^3 |x-2| dx$.

(c) $\int \frac{1}{x^2} \sec\left(\frac{1}{x}\right) \tan\left(\frac{1}{x}\right) dx$

(d) $\int \frac{\sin 3x - \cos 3x}{\sin 3x + \cos 3x} dx$

(e) $\int \frac{1}{\ln(x^x)} dx$

(f) $\int \frac{e^{2x}}{\sqrt{4-3e^{2x}}} dx$

(g) $\int \frac{dx}{\sqrt{4-(x+3)^2}}$

(h) $\int \frac{5p^{\sqrt{x+1}}}{\sqrt{x+1}} dx, \quad p > 0$

2. Find the area bounded between the curves.

(a) $y = 2\cos x$ and $y = \sin(2x)$ on the interval $[-\pi, \pi]$.

(b) $y = -x^3 - 2x^2 + 7x - 2$ and $y = -x - 2$.

(c) $f(x) = x^3 + 2x^2$ and $g(x) = x^2 + 2x$.

(d) $y = x^3$, $y = -x$, and $y = -1$.

3. Find the area of the triangle with vertices at the points (0,1), (3,4), and (4,2). USE CALCULUS.

4. Evaluate the integrals:

(a) $\int \left(\sqrt{x} - \frac{1}{x^2}\right)^2 dx$

(b) $\int \frac{\log_3 x^4}{x} dx$

(c) $\int \frac{\sec(e^{-4x})}{e^{4x}} dx$

(d) $\int_1^e \frac{\sqrt{\ln x}}{x} dx$

(e) $\int \frac{x^2}{(ax^3+b)^2} dx$

(f) $\int_{-5}^0 \left(x\sqrt{4-x}\right) dx$

(g) $\int (1+\ln x)\cot(x\ln x)dx$

(h) $\int_0^{\ln(3)} e^{-2x}dx$

(i) $\int_0^8 \frac{x}{\sqrt{1+x}}dx$

(j) $\int_0^{\pi/4} \left(\frac{\sec x}{\tan x+1}\right)^2 dx.$

5. As you go trick-or-treating, you walk with a velocity of $v(t) = \frac{e^t}{1+e^t}$ feet per minute. How far have you traveled after $\ln 5$ minutes?

6. Evaluate each integral below using any of the methods we have learned.

(a) $\int \frac{\sin^3 x}{\cos x}dx$

(b) $\int \frac{x}{\sqrt{x^2+2x-3}}dx$

(c) $\int \frac{\cos x}{4+\sin^2 x}dx$

(d) $\int \frac{1}{x(x^2+x+1)}dx$

(e) $\int 3x\cos(2x)dx$

(f) $\int x^5 \ln(x)dx$

(g) $\int x^3 e^{x^2}dx$

(h) $\int (\ln x)^2 dx$

(i) $\int x^2 \cdot 4^x dx$

(j) $\int \cos(2x)e^x dx$

7. Evaluate each improper integral if it converges, or show that the integral diverges.

(a) $\int_0^3 \frac{x}{(x^2-1)^{2/3}}dx$

(b) $\int_1^3 \frac{1}{(x^2-1)^{3/2}}dx$

(c) $\int_0^\infty x^2 e^{-2x}dx$

(d) $\int_1^\infty \frac{1-\ln(x)}{x^2}dx$

8. Evaluate the following integrals using any of the integration techniques we have learned.

(a) $\int \sin^5(2x)\cos^3(2x)dx$

(b) $\int_1^e \frac{\sqrt{\ln x}}{x}dx$

(c) $\int \tan^4(x)dx$

(d) $\int \frac{x^2}{(x^2+4)^{3/2}}dx$

(e) $\int \frac{\sqrt{1-x^2}}{x^4}dx$

(f) $\int \frac{x}{(4-x^2)^{3/2}}dx$

(g) $\int \frac{dx}{e^x\sqrt{e^{2x}-9}}$

(h) $\int \sin^2(x)\cos^2(x)dx$

(i) $\int (x^2+1)e^{2x}dx$

(j) $\int \frac{x+3}{(x-1)(x^2-4x+4)}dx$

(k) $\int \frac{x+4}{x^3+x}dx$

(l) $\int \tan(x)\ln[\cos(x)]dx$

(m) $\int \frac{x+2}{x+1}dx$

(n) $\int \sqrt{25-x^2}dx$

(o) $\int \tan^3(x)\sec^4(x)dx$

(p) $\int x\tan^{-1}(x)dx$

(q) $\int \frac{dx}{x\sqrt{1+x^2}}$

(r) $\int \frac{x+1}{x^2(x-1)}dx$

(s) $\int \frac{x+1}{x^2-4x+8}dx$

9. For what values of p does the integral converge?

$$\int_4^\infty \frac{dx}{x(\ln x)^p}$$

10. Find the area bounded by the curve $y = \frac{1}{x^2+9}$, the x-axis, and $x \geq 0$.

11. Evaluate the following integrals using any method we have learned.

(a) $\int \frac{1}{x^2\sqrt{1+x^2}}dx$

(b) $\int \frac{x^5}{\sqrt{4-x^2}}dx$

(c) $\int \tan^3(x)\sec^3 x dx$

(d) $\int \cot(x)\sec^2 x dx$

(e) $\int \frac{\sin^7(x)}{\cos^4(x)}dx$

(f) $\int \sec^4(x)dx$

(g) $\int \frac{x^3-1}{x^2+1}dx$

(h) $\int \frac{\cos(2x)}{\sin^2(2x)-3\sin(2x)-4}dx$

12. Use series to write the repeating decimals (a) 0.31313131... and (b) $0.3\overline{27}$ as rational numbers.

13. **Find the sum** of each convergent series below, or explain why the series diverges.

(a) $\sum_{k=7}^{\infty} \frac{1}{(k-3)(k+1)}$

(b) $\sum_{k=0}^{\infty}(-1)^k$

(c) $\sum_{k=2}^{\infty} \frac{2^k+1}{3^{k+1}}$

(d) $\sum_{n=2}^{\infty} \frac{2^{2n-1}}{4\cdot 10^{n-1}}$

(e) $\sum_{n=2}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

(f) $\sum_{n=1}^{\infty} \frac{3n+4}{n^3+3n^2+2n}$

14. Determine if each series below converges or diverges. JUSTIFY YOUR ANSWER FULLY using any of the convergence tests from class.

(a) $\sum_{k=1}^{\infty} \frac{e^k}{4+e^{2k}}$

(b) $\sum_{k=1}^{\infty} \frac{5k^2+8}{7k^2+6k+1}$

(c) $\sum_{k=1}^{\infty} \frac{3^{2k}}{8^k-3}$

(d) $\sum_{k=1}^{\infty} \frac{k+2}{\sqrt{k^5+4}}$

(e) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$

(f) $\sum_{k=1}^{\infty} k \tan\left(\frac{1}{k}\right)$

(g) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$

(h) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

(i) $\sum_{k=1}^{\infty} \frac{\ln k}{k^4}$

(j) $\sum_{k=1}^{\infty} \frac{(2k)^k}{k!}$

(k) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{2k^2}$

(l) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{4^n 2^n n!}$

(m) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{n^{2n}}$

(n) $\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1+\frac{1}{2n}\right)^{n^2}$

(o) $\sum_{n=1}^{\infty} \frac{2^n+n^2}{3^n+n}$

(p) $\sum_{n=2}^{\infty} \frac{(1+(-1)^n)n^2}{(n-1)^3}$

(q) $\sum_{k=1}^{\infty} k^3 e^{-k^2}$

15. Suppose $r > 0$. Find the values of r, if any, for which $\sum_{k=1}^{\infty} \frac{r^k}{k^r}$ converges.

16. Determine whether the following alternating series converge absolutely, converge conditionally, or diverge. Justify your answers using the tests we discussed in class.

(a) $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{3k}{\sqrt{k^3+4}}$

(b) $\sum_{k=2}^{\infty} (-1)^k \frac{k}{k^4-1}$

(c) $\sum_{k=0}^{\infty} (-1)^k \frac{k}{5^k+2^k}$

(d) $\sum_{k=3}^{\infty} (-1)^k \frac{1}{k \ln k \sqrt{\ln \ln k}}$

(e) $\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1}$

(f) $\sum_{n=3}^{\infty} (-1)^n \frac{n-2}{n+2}$

(g) $\sum_{n=4}^{\infty} (-1)^n \frac{1}{n-3}$

(h) $\sum_{n=0}^{\infty}(-5)^{-n}$

(i) $\sum_{n=1}^{\infty}(-1)^n \frac{\ln(n)}{\sqrt{n}}$

(j) $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^2-n}$

17. Find the radius and interval of convergence of the following power series:

(a) $\sum_{k=2}^{\infty}\left(\frac{k}{k-1}\right)\frac{(x+2)^k}{2^k}$

(b) $\sum_{k=1}^{\infty}\frac{k}{3^k(k^2+1)}(x-5)^k$

(c) $\sum_{k=1}^{\infty}\frac{5^k}{\sqrt{k}}(3-2x)^k$

(d) $\sum_{k=1}^{\infty}\frac{(-1)^k a^k}{k^2}(x-a)^k$, where $a \neq 0$

18. Find all p such that the infinite series $\sum_{n=3}^{\infty}\frac{1}{n^p \ln(n)}$ converges.

19. Consider the power series $f(x) = \sum_{n=0}^{\infty}\left(\frac{x^2+2}{6}\right)^n$.

(a) Find the interval of convergence for this series.

(b) For all values of x that lie in the interval of convergence, what is the sum of this series as a function of x?

20. Determine how many terms must be used to determine the sum of the entire series with an error of less than 0.01

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+1}$

(b) $\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n$

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

21. For which value of p does the following series converge to 7?

$$\sum_{k=0}^{\infty}\frac{3}{2^{pk}}$$

22. Find the third degree Taylor polynomial of the function $f(x) = \tan^{-1}(x)$ in powers of $x-1$.

23. Use a Taylor polynomial to estimate the value of $\sqrt{e}$ with an error of at most 0.01. HINT: Choose $a = 0$ and use the fact that $e < 3$.

24. Use the MacLaurin series for $f(x) = \frac{1}{1-x}$ to find a power series representation of the function

$$g(x) = \frac{x}{(1-x)^3}.$$

HINT: You will need to differentiate.

25. Let $f(x) = \int_0^x t\sin(t^3)dt$. Use a MacLaurin series to find $f^{(11)}(0)$.

26. Find the sum of the series:

$$1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} + ... + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + ...$$

27. For what values of x can we replace $\cos x$ with $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ within an error range of no more that 0.001?

28. Find $f^{(7)}(0)$ for the function $f(x) = x\sin(x^2)$.

29. Find a power series representation for the following functions. When is your series valid?

(a) $f(x) = \frac{3x}{2+4x}$

(b) $g(x) = \int_0^x \frac{\sin(t/2)}{2t}dt$

(c) $h(x) = \tan^{-1}(x)$

30. (a) Estimate $\cos 15°$ using a fourth-degree Taylor polynomial.

(b) Estimate $\int_0^1 e^{-2x^2}dx$ within an error of 0.01.

31. Find the volume of the solid generated when rotating the region:

(a) bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis.

(b) bounded by the curves $y = x^2 - 4$ and $y = 2x - x^2$ about the line $y = -4$.

(c) bounded by the curves $x = y^2 + 1$ and $x = 3$ about the line $x = 3$

(d) bounded by the curves $y = \sin x \cos x$, $x = 0$, and $x = \frac{\pi}{2}$ about the x-axis.

32. Find the volume of the solid generated by revolving the triangular region with vertices at $(1, 0)$, $(2, 1)$, and $(1, 1)$ about the y-axis.

Answers

1. (a) $-\frac{3}{8}$, (b) 1

(c) $-\sec\left(\frac{1}{x}\right) + C$

(d) $-\frac{1}{3}\ln|\sin 3x + \cos 3x| + C$

(e) $\ln|\ln x| + C$

(f) $-\frac{1}{3}\sqrt{4 - 3e^{2x}}$

(g) $\sin^{-1}\left(\frac{x+3}{2}\right) + C$

(h) $\frac{10}{\ln p} p^{\sqrt{x+1}} + C$

2. (a) 8, (b) $\frac{148}{3}$, (c) $\frac{37}{12}$, (d) $\frac{5}{4}$ all in square units

3. 4.5 square units

4. (a) $\frac{1}{2}x^2 + \frac{4}{\sqrt{x}} - \frac{1}{3x^3} + C$

(b) $2\ln 3(\log_3 x)^2 + C$

(c) $-\frac{1}{4}\ln|\sec(e^{-4x}) + \tan(e^{-4x})| + C$

(d) $\frac{2}{3}$

(e) $-\frac{1}{3a(ax^3+b)} + C$

(f) $-\frac{506}{15}$

(g) $\ln|\sin(x\ln x)| + C$

(h) $\frac{4}{9}$, (i) $\frac{40}{3}$, (j) $\frac{1}{2}$

5. $\ln 3$ feet

6. (a) $-\ln|\cos x| + \frac{1}{2}\cos^2 x + C$

(b) $\sqrt{x^2 + 2x - 3} - \ln|\frac{x+1+\sqrt{x^2+2x-3}}{2}| + C$

(c) $\frac{1}{2}\tan^{-1}\left(\frac{\sin x}{2}\right) + C$

(d) $\ln|x| - \frac{1}{2}\ln(x^2 + x + 1) - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

(e) $\frac{3}{2}x\sin(2x) + \frac{3}{4}\cos(2x) + C$

(f) $\frac{x^6\ln x}{6} - \frac{x^6}{36} + C$

(g) $\frac{x^2e^{x^2}}{2} - \frac{e^{x^2}}{2} + C$

(h) $x(\ln x)^2 - 2x\ln x + 2x + C$

(i) $\frac{1}{\ln 4}x^2 \cdot 4^x - \frac{2}{(\ln 4)^2}x \cdot 4^x + \frac{2}{(\ln 4)^3}4^x + C$

(j) $\frac{1}{5}\cos(2x)e^x + \frac{2}{5}\sin(2x)e^x + C$

7. (a) Converges to $\frac{9}{2}$, (b) diverges, (c) converges to $\frac{1}{4}$, (d) converges to 0

8. (a) $\frac{1}{12}\sin^6(2x) - \frac{1}{16}\sin^8(2x) + C$

(b) $\frac{2}{3}$

(c) $\frac{1}{3}\tan^3(x) - \tan(x) + x + C$

(d) $\ln|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}| - \frac{x}{\sqrt{x^2+4}} + C$

(e) $-\frac{1}{3} \cdot \frac{(1-x^2)^{3/2}}{x^3} + C$

(f) $\frac{1}{\sqrt{4-x^2}} + C$

(g) $\frac{\sqrt{e^{2x}-9}}{9e^x} + C$

(h) $\frac{x}{8} - \frac{1}{32}\sin(4x) + C$

(i) $\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

(j) $4\ln\left|\frac{x-1}{x-2}\right| - \frac{5}{x-2} + C$

(k) $4\ln|x| - 2\ln(x^2+1) + \tan^{-1}(x) + C$

(l) $-\frac{1}{2}(\ln[\cos(x)])^2 + C$

(m) $x + \ln|x+1| + C$

(n) $\frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x\sqrt{25-x^2}}{2} + C$

(o) $\frac{1}{4}\tan^4(x) + \frac{1}{6}\tan^6(x) + C$

(p) $\frac{x^2}{2}\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$

(q) $-\ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C$

(r) $-2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

(s) $\frac{1}{2}\ln|x^2 - 4x + 8| + \frac{3}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$

9. converges when $p > 1$

10. $\frac{\pi}{6}$ units^2

11. (a) $\frac{\sqrt{1+x^2}}{x} + C$

(b) $-32\left(\left(\frac{\sqrt{4-x^2}}{2}\right) - \frac{2}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5\right) + C$

(c) $-\frac{1}{3}\sec^3(x) + \frac{1}{5}\sec^5(x) + C$

(d) $\ln(\tan(x)) + C$

(e) $\frac{1}{3\cos^3(x)} - \frac{3}{\cos(x)} - 3\cos(x) + \frac{1}{3}\cos^3(x) + C$

(f) $(2\tan(x))/3 + 1/3\sec(x)^2\tan(x) + C$

(g) $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) + C$

(h) $-\frac{1}{10}\ln|\sin(2x)+1| + \frac{1}{10}\ln|\sin(2x)-4| + C$

12. (a) $\frac{31}{99}$, (b) $\frac{108}{330}$

13. (a) ≈ 0.1899, (b) diverges, (c) $\frac{1}{2}$, (d) $\frac{1}{3}$, (e) $\frac{3}{4}$, (f) $\frac{5}{2}$

14. (a) converges by integral test, (b) diverges by divergence test, (c) diverges by direct comparison, (d) converges by direct or limit comparison, (e) converges by integral test, (f) diverges by divergence test, (g) converges by direct comparison, (h) converges by limit comparison, (i) converges by direct comparison, (j) diverges by ratio test, (k) converges by root test, (l) converges by ratio test, (m) converges by ratio test, (n) converges by root test, (o) converges by limit comparison test, (p) diverges by direct comparison, (q) converges by the integral test

15. converges when $0 < r < 1$

16. (a) converges conditionally (limit comparison and alternating series test), (b) converges absolutely (limit comparison), (c) converges absolutely (ratio test), (d) converges conditionally (integral test and alternating series test), (e) conditionally convergent (direct comparison and alternating series test), (f) diverges (nth term test), (g) conditionally convergent (direct comparison and alternating series test), (h) absolutely convergent (geometric), (i) conditionally convergent (integral test and alternating series test), (j) absolutely convergent (limit comparison test)

17. (a) $R = 2$, $I.C. = (-4, 0)$, (b) $R = 3$, $I.C. = [2, 8)$ (c) $R = \frac{1}{10}$, $I.C. = \left(\frac{7}{5}, \frac{8}{5}\right]$, (d) $R = \frac{1}{|a|}$, $I.C. = [a - \frac{1}{|a|}, a + \frac{1}{|a|}]$

18. converges for $p > 1$, diverges for $p \leq 1$

19. $I.C. = (-2, 2)$, sum is $\frac{6}{4-x^2}$

20. (a) 9, (b) 4, (c) 100

21. $p = \log_2(7) - 2 \approx 0.8074$

22. $P_3(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3$

23. $\sqrt{e} \approx f(0.5) = 1 + 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} = 1.6458.$

24. $\frac{1}{2}\sum_{k=2}^{\infty} k(k-1)x^{k-1}$, $|x| < 1$

25. $-\frac{10!}{6}$

26. $\frac{1}{2}$

27. $x \in (-0.9467, 0.9467)$

28. -840

29. (a) $3\sum_{k=0}^{\infty}(-1)^k 2^{k-1}x^{k+1}$, valid for $|x| < \frac{1}{2}$
(b) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2^{2k+2}(2k+1)!(2k+1)}$, valid for $x \in \Re$
(c) $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} + C$, valid for $|x| < 1$

30. (a) 0.966; (b) approximately 0.6165 (stop at $k = 5$)

31. (a) $\frac{108\pi}{5}$, (b) 45π, (c) $\frac{64\pi\sqrt{2}}{15}$, (d) $\frac{\pi^2}{16}$ (all in cubic units)

32. $\frac{4\pi}{3}$ cubic units

FINAL EXAM FORMULA SHEET

The following formulas will be provided for you on the final exam. You will need to memorize any formulas needed to work the problems that are not on this list.

$$\sin(2x) = 2\sin x \cos x$$

$$\sin^2 x = \frac{1}{2}\left[1 - \cos(2x)\right]$$

$$\cos^2 x = \frac{1}{2}\left[1 + \cos(2x)\right]$$

$$\int \tan(x)dx = \ln|\sec(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)dx = -\ln|\csc(x) + \cot(x)| + C$$

$$\int b^x dx = \frac{1}{\ln b} b^x + C$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$|R_n(x)| \leq \frac{\max|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in (-\infty, \infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad x \in (-\infty, \infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in (-\infty, \infty)$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad x \in (-1, 1)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, \quad x \in (-1, 1]$$